A monoidal representation for linearized gravity

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- Monoidal Space
- Linearized Gravity
- Monoidal Representation

Monoidal Space

Let γ be a parametric curve in $\mathbb{R}^3$

$$I^{ab}[\vec{x},\gamma] := \int_{\gamma} dl u^a_{T_\gamma} u^b_{T_\gamma} \delta^3(\vec{z}_\gamma - \vec{x}); \qquad a,b = 1,2,3.$$

Let us consider $\mathcal{M} = \{\Gamma : \Gamma = \gamma_1 \cup .... \cup \gamma_n\}$, then

$$I^{ab}[\vec{x},\Gamma] = I^{ab}[\vec{x},\gamma_1 \cup ... \cup \gamma_n] = \sum_{i=1}^{n} I^{ab}[\vec{x},\gamma_i]$$

$$\Gamma \approx \Gamma' \Leftrightarrow I^{ab}[\vec{x},\Gamma] = I^{ab}[\vec{x},\Gamma']$$

Monoidal Space

- Reflexive. $\Gamma \approx \Gamma$

$$I^{ab}[\vec{x}, \Gamma] = I^{ab}[\vec{x}, \Gamma]$$

- Symmetric. $\Gamma \approx \Gamma' \Rightarrow \Gamma' \approx \Gamma$

$$I^{ab}[\vec{x}, \Gamma] = I^{ab}[\vec{x}, \Gamma'] \Rightarrow I^{ab}[\vec{x}, \Gamma'] = I^{ab}[\vec{x}, \Gamma]$$

- Transitive. $\Gamma \approx \Gamma'$ and $\Gamma' \approx \Gamma''$, then $\Gamma \approx \Gamma''$

  $I^{ab}[\vec{x}, \Gamma] = I^{ab}[\vec{x}, \Gamma']$ and $I^{ab}[\vec{x}, \Gamma'] = I^{ab}[\vec{x}, \Gamma'']$

$$\Rightarrow I^{ab}[\vec{x}, \Gamma] = I^{ab}[\vec{x}, \Gamma'']$$

$\Rightarrow \Gamma, \Gamma'$ **represent the same path belonging to** $\mathfrak{M} = \mathcal{M}/R$

Monoidal Space

We define a multiplication rule $\circ$ for elements in $\mathfrak{M}$ as

$$\begin{aligned} I^{ab}[\vec{x},\Gamma_1\circ\Gamma_2] &:= I^{ab}[\vec{x},\Gamma_1\cup\Gamma_2] = I^{ab}[\vec{x},\Gamma_1] + I^{ab}[\vec{x},\Gamma_2] \\ &= I^{ab}[\vec{x},\Gamma_2] + I^{ab}[\vec{x},\Gamma_1] = I^{ab}[\vec{x},\Gamma_2\circ\Gamma_1] \end{aligned}$$

Properties

- Closure. If $\Gamma_1,\Gamma_2\in\mathfrak{M}$ then $\Gamma_1\circ\Gamma_2\in\mathfrak{M}$
- Assosiative $\Gamma_1\circ(\Gamma_2\circ\Gamma_3) = (\Gamma_1\circ\Gamma_2)\circ\Gamma_3$

$$\begin{aligned} I^{ab}[\vec{x},\Gamma_1\circ(\Gamma_2\circ\Gamma_3)] &= I^{ab}[\vec{x},\Gamma_1] + (I^{ab}[\vec{x},\Gamma_2] + +I^{ab}[\vec{x},\Gamma_3]) \\ &= (I^{ab}[\vec{x},\Gamma_1] + I^{ab}[\vec{x},\Gamma_2]) + I^{ab}[\vec{x},\Gamma_3] \\ &= I^{ab}[\vec{x},(\Gamma_1\circ\Gamma_2)\circ\Gamma_3] \end{aligned}$$

- Neutral Element For a path Γ_e such as $I^{ab}[\Gamma_e] = 0 \Rightarrow I^{ab}[\vec{x},\Gamma\circ\Gamma_e] = I^{ab}[\vec{x},\Gamma]$

$\mathfrak{M}\rightarrow$ **COMMUTATIVE MONOIDE**

Monoidal Space

Let us consider path valued functionals $\Psi : \mathfrak{M} \to \mathbb{C}$.

$$v^c v^d \delta_{cd}(\vec{y},\hat{v})\Psi[\Gamma] := \lim_{L\to 0} \frac{1}{L}\left(\Psi[\Gamma\circ\Gamma[\vec{y},\hat{v}]] - \Psi[\Gamma]\right)$$

$\delta_{cd}(\vec{y},\hat{v})$ path derivative operator

Now,

$$\begin{aligned}
v^c v^d \delta_{cd}(\vec{y},\hat{v}) I^{ab}[\vec{x},\Gamma] &= \lim_{L\to 0} \frac{1}{L}\left(I^{ab}[\vec{x},\Gamma\circ\Gamma[\vec{y},\hat{v}]] - I^{ab}[\vec{x},\Gamma]\right) \\
&= \lim_{L\to 0} \frac{1}{L}\left(\cancel{I^{ab}[\vec{x},\Gamma]} + I^{ab}[\vec{x},\Gamma[\vec{y},\hat{v}]] - \cancel{I^{ab}[\vec{x},\Gamma]}\right) \\
&= \lim_{L\to 0} \frac{1}{L} I^{ab}[\vec{x},\Gamma[\vec{y},\hat{v}]] \\
&= \lim_{L\to 0} \frac{1}{\cancel{L}} \cancel{L} v^a v^b \delta^3(\vec{x}-\vec{y}) = v^a v^b \delta^3(\vec{x}-\vec{y})
\end{aligned}$$

$$\Rightarrow \delta_{cd}(\vec{y},\hat{v}) I^{ab}[\vec{x},\Gamma] = \frac{1}{2}(\delta^a_c\delta^b_d + \delta^a_d\delta^b_c)\delta^3(y-x)$$

Linearized Gravity

Canonical Variables

$$(h_{ab}, p^{ab})$$

Poisson Algebra

$$\{h_{ab}(\vec{x}), p^{cd}(\vec{y})\} = \frac{1}{2}(\delta^c_a \delta^d_b + \delta^c_b \delta^d_a)\delta^3(\vec{x} - \vec{y})$$

First Class Contraints

$$\begin{aligned} \partial^a \partial^b h_{ab} - \partial^a \partial_a h &\approx 0 \\ \partial_a p^{ab} &\approx 0 \end{aligned}$$

Linearized Gravity

Dirac quantization program

$$\begin{aligned} h_{ab} &\rightarrow \hat{h}_{ab} \\ p^{ab} &\rightarrow \hat{p}^{ab} \end{aligned}$$

$$\begin{aligned} [\hat{h}_{ab}(\vec{x}), \hat{p}^{cd}(\vec{y})] &\rightarrow i\{h_{ab}(\vec{x}), p^{cd}(\vec{y})\} \\ &= \frac{i}{2}(\delta^c_a \delta^d_b + \delta^c_b \delta^d_a)\delta^3(\vec{x} - \vec{y}). \end{aligned}$$

$$\begin{aligned} \partial_a \hat{p}^{ab}|\Psi\rangle_{fis} &\approx 0 \\ (\partial^a \partial^b \hat{h}_{ab} - \partial^a \partial_a \hat{h})|\Psi\rangle_{fis} &\approx 0 \end{aligned}$$

Monoidal Representation

$$
\begin{array}{rcl}
\hat{h}_{ab}\Psi[\Gamma] & \rightarrow & i\delta_{ab}(\vec{x},\hat{v})\Psi[\Gamma] \\
\hat{p}^{ab}\Psi[\Gamma] & \rightarrow & I^{ab}[\vec{x},\Gamma]\Psi[\Gamma]
\end{array}
$$

Now

$$
\begin{array}{rcl}
[i\delta_{ab}(\vec{x},\hat{v}), I^{ab}[\vec{y},\Gamma]]\Psi[\Gamma] & = & i\delta_{ab}(\vec{x},\hat{v})(I^{cd}[\vec{y},\Gamma]\Psi[\Gamma]) \\
& - & iI^{cd}[\vec{y},\Gamma]\delta_{ab}(\vec{x},\hat{v})\Psi[\Gamma] \\
& = & (\delta_{ab}(\vec{x},\hat{v})I^{cd}[\vec{y},\Gamma])\Psi[\Gamma] \\
& = & \frac{i}{2}(\delta^c_a\delta^d_b + \delta^c_b\delta^b_a)\delta^3(\vec{x}-\vec{y})\Psi[\Gamma]
\end{array}
$$

Fulfills the canonical algebra!

But...do not solve the first class constraints automatically...

Monoidal Representation

Reduced phase space quantization

$$
\begin{aligned}
\hat{h}^{TT}_{ab} &= P_{ab/cd}\hat{h}_{cd} \\
\hat{p}^{abTT} &= P_{ab/cd}\hat{p}^{cd}
\end{aligned}
$$

where

$$P_{ab/cd} := P_{ac}P_{db} - \frac{1}{2}P_{ab}P_{cd}$$

and

$$P_{ab} = \delta_{ab} - \frac{1}{2}\partial_a\partial_b\nabla^{-2}$$

Geometrical interpretation????

Monoidal Representation

Abelian Loop Representation: source-free Electromagnetism

$$T^a[\vec{x}, \gamma] := \int_\gamma dy^a \delta^3(\vec{x} - \vec{y})$$

Loop representation

$$\begin{aligned} \hat{E}^a \Psi[\gamma] &\rightarrow T^a[\vec{x}, \gamma] \Psi[\Gamma] \\ \hat{A}_a \Psi[\gamma] &\rightarrow \delta_a(\vec{x}) \Psi[\gamma] \end{aligned}$$

Gauss Constraint

$$\partial_a \hat{E}^a \Psi[\gamma] \approx 0$$

Automatically solved!

Monoidal Representation

Skein Representation: Linearized Gravity[1]

$$
\begin{aligned}
\hat{h}_{ab}(\vec{x})\Psi[\gamma] &\rightarrow \frac{1}{2\sqrt{2}} i(\delta_a[\vec{x},\gamma_b] + \delta_b[\vec{x},\gamma_a])\Psi[\Gamma] \\
\hat{p}^{ab}(\vec{x})\Psi[\gamma] &\rightarrow \frac{1}{2\sqrt{2}} (T^a[\vec{x},\gamma_b] + T^b[\vec{x},\gamma_a])\Psi[\gamma]
\end{aligned}
$$

Geometrical Interpretation????

[1] EC and Lorenzo Leal. Int. J. Mod. Phys. D 23, 1450047

Monoidal Representation

Generator of duality rotations

$$\hat{G} = \int d^3x\left(-\frac{1}{4}\hat{h}^{TT}_{ab}\,\varepsilon^{acd}\partial_c\hat{h}^{TT}_{db} + \hat{p}^{TT}_{ab}\nabla^{-2}\varepsilon^{acd}\partial_c\hat{p}^{TT}_{db}\right).$$

" Metric dependent Linking Number"

$$\oint_{\Gamma_1} dl_{\Gamma_1} \oint_{\Gamma_2} dl_{\Gamma_2}\, \hat{u}_{T_{\Gamma_1}} \times \hat{u}_{T_{\Gamma_2}} \cdot \frac{(\vec{z}_{\Gamma_1} - \vec{z}_{\Gamma_2})}{|\vec{z}_{\Gamma_1} - \vec{z}_{\Gamma_2}|^3}(\hat{u}_{T_{\Gamma_1}} \cdot \hat{u}_{T_{\Gamma_2}})$$

Monoidal Representation

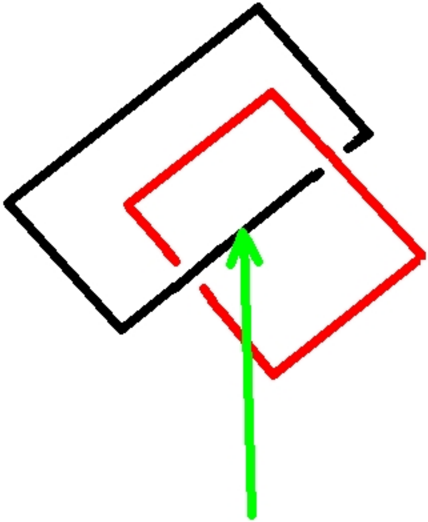
If we consider

$$-\frac{1}{4\pi}\frac{(\vec{x}-\vec{y})^a}{|\vec{x}-\vec{y}|^3} = \int_{\gamma}^{\vec{y}} dz^a \delta^3(\vec{x}-\vec{y}) + \varepsilon^{abc}\partial_b f_c(|\vec{x}-\vec{y}|)$$

The metric dependent linking number takes the form...

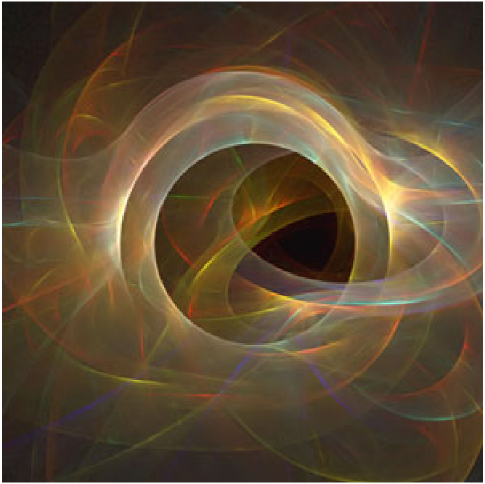
$$\oint_{\Gamma_1} dl_{\Gamma_2} \oint_{\Gamma_2} dl_{\Gamma_2}(\hat{u}_{T_{\Gamma_1}} \cdot \hat{u}_{T_{\Gamma_2}}) \int_{\gamma}^{\vec{z}_{\Gamma_1}} dl_{\gamma}(\hat{u}_{T_{\Gamma_1}} \times \hat{u}_{T_{\Gamma_2}}) \cdot \hat{u}_{T_\gamma} \delta^3(\vec{z}_{\Gamma_2} - \vec{w}_\gamma)$$

Monoidal Representation



Conclusions

- Linearized Gravity admits a monoidal representation in the reduced phase space.
- Generator of duality admits a geometrical interpretation in terms of a "metric dependent" Gauss Linking Number.



Gracias!