From Scalar Galileons to Generalized and Covariantized (non-Abelian) Vector Galileons

Erwan Allys (IAP), Juan P. Beltrán Almeida (UAN), Patrick Peter (IAP), Yeinzon Rodríguez (UAN, UIS, ICTP)

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Motivation: the Ostrogradsky's instability

* Why are most of the laws in physics represented by second-order differential equations?

* Examples:
* Newton's second law.
* Maxwell's equations.
* Einstein's field equations.

* The answer relies on the Ostrogradsky's instability (**Mem. Ac. St. Petersburg 1850**)
* Let's think first of a mechanical system with just one degree of freedom:

$$L = L(q, \dot{q}) .$$

* The Euler-Lagrange equation leads to a second-order differential equation:

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0 ,$$

as long as the non degeneracy condition is satisfied, i.e., $\dfrac{\partial L}{\partial \dot{q}}$ must depend on $\dot{q}$.

* The non degeneracy condition is equivalent to saying that $\dot{q}$ does not disappear in the Lagrangian by partial integrations.
* Thus, $\ddot{q} = \mathcal{F}(q, \dot{q}) \longrightarrow q(t) = \mathcal{Q}(t, q_0, \dot{q}_0)$.
* Let's look for the Hamiltonian: the two canonical variables are usually taken as

$$Q \equiv q, \quad P \equiv \frac{\partial L}{\partial \dot{q}} \quad ,$$

* The non degeneracy condition guarantees that $\dot{q}$ can be written in terms of Q and P:

$$\dot{q} = v(Q, P)$$

* Therefore, $H(Q,P) \equiv P\dot{q} - L$

$$= Pv(Q,P) - L(Q,v(Q,P)) \ .$$

* With these canonical variables, the canonical evolution equations reproduce the inverse phase space transformation and the Euler-Lagrange equation:

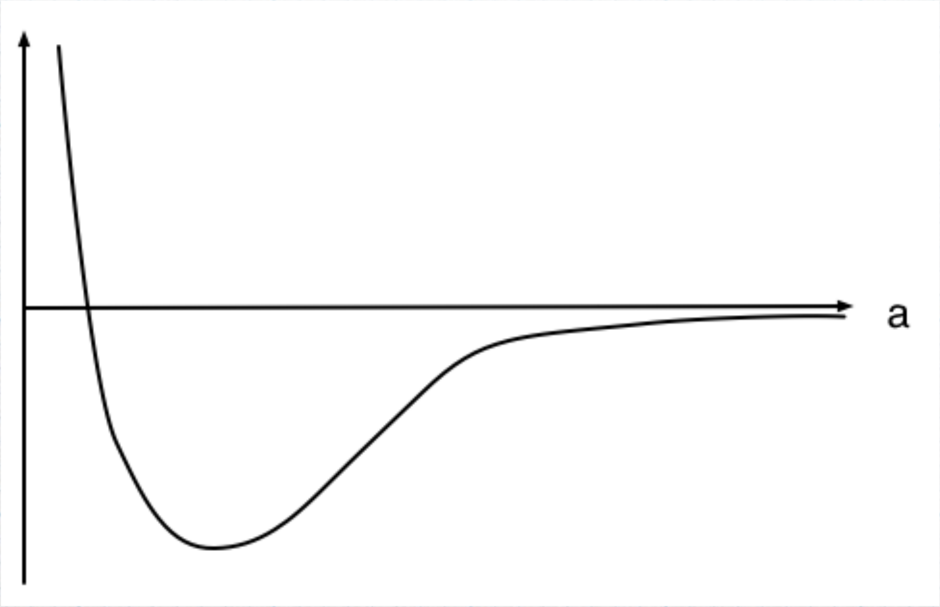
$$\dot{Q} = \frac{\partial H}{\partial P} = v(Q,P) \ ,$$

$$\dot{P} = -\frac{\partial H}{\partial Q} = \frac{\partial L}{\partial q} \ .$$

* We can conclude then that the Hamiltonian is the energy up to canonical transformations.

- Observing the Hamiltonian
$$H(Q,P) = Pv(Q,P) - L(Q, v(Q,P))$$
we conclude that it, in principle, isn't linear either in Q or in P.

- It, in principle, doesn't suffer from the Ostrogradsky's instability (the energy is, in principle, bounded from below).



* What about if the Lagrangian depends also on $\ddot{q}$?

* The equation of motion is, therefore, higher than second order

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} + \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{q}} = 0,$$

$$q^{(4)} = \mathcal{F}(q, \dot{q}, \ddot{q}, q^{(3)}) \longrightarrow q(t) = \mathcal{Q}(t, q_0, \dot{q}_0, \ddot{q}_0, q_0^{(3)}),$$

as long as the non degeneracy condition is satisfied, i.e., $\dfrac{\partial L}{\partial \ddot{q}}$ must depend on $\ddot{q}$.

* Let's look for the Hamiltonian. We require four canonical variables. The Ostrogradsky's choice is the following:

$$Q_1 \equiv q, \quad P_1 \equiv \frac{\partial L}{\partial \dot{q}} - \frac{d}{dt}\frac{\partial L}{\partial \ddot{q}},$$

$$Q_2 \equiv \dot{q}, \quad P_2 \equiv \frac{\partial L}{\partial \ddot{q}}.$$

* The non degeneracy condition guarantees that $\ddot{q}$ can be written in terms of Q_1, Q_2 and P_2 :

$$\ddot{q} = a(Q_1, Q_2, P_2)$$

* The Hamiltonian is therefore

$$H(Q_1, Q_2, P_1, P_2) \equiv \sum_{i=1}^{2} P_i q^{(i)} - L$$

$$= P_1 Q_2 + P_2 a(Q_1, Q_2, P_2) - L(Q_1, Q_2, a(Q_1, Q_2, P_2))$$

* As before, the canonical evolution equations reproduce the inverse phase space transformations and the Euler-Lagrange equation:

$$\dot{Q}_1 = \frac{\partial H}{\partial P_1} = Q_2 \qquad \dot{Q}_2 = \frac{\partial H}{\partial P_2} = a(Q_1, Q_2, P_2)$$

$$\dot{P}_1 = -\frac{\partial H}{\partial Q_1} = \frac{\partial L}{\partial q} \qquad \dot{P}_2 = -\frac{\partial H}{\partial Q_2} = -P_1 + \frac{\partial L}{\partial \dot{q}}$$

* We can conclude then that the Hamiltonian is the energy up to canonical transformations.

* Observing the Hamiltonian

$$H(Q_1, Q_2, P_1, P_2)$$
$$= P_1 Q_2 + P_2 a(Q_1, Q_2, P_2) - L(Q_1, Q_2, a(Q_1, Q_2, P_2))$$

we observe that it is linear in P_1 .

* Very bad!!: it suffers from the Ostrogradsky's instability (the energy is not bounded from below - see the blue curve).

* It seems to be that there's no any other way: equations of motion must be second order!
* What happens if we include more derivatives in the Lagrangian?: they only aggravate the problem.

Scalar Galileons

* Motivation: bottom-up approach to fundamental physics (and to do cosmology as well!).
* G. W. Horndeski in 1974 (Int. J. Theor. Phys. 1974) was able to obtain the most general scalar-tensor theory with second-order field equations in curved four-dimensional spacetime.
* Horndeski was largely ignored until 2009 when his work was rediscovered in the context of what is now called Galileons.
* When Horndeski, in 1981, took a sabbatical year in the Netherlands, he saw a van Gogh exhibition. He was so deeply moved that he left physics and became an artist.
* Nowadays, Horndeski's work in mathematical physics is highly cited and his artwork is highly appreciated (he continues doing some physics).

* Galileons were introduced by Nicolis et. al. (Phys. Rev. D 2009) inspired from the decoupling limit of the Dvali-Gabadadze-Porrati model.

* Galileons are those scalar fields π in flat spacetime whose

1. Lagrangian is degenerate but, still, contains derivatives of π of order 2 or less.

2. field equations are polynomial in second-order derivatives of π .

3. field equations do not contain undifferentiated or only once differentiated π .

4. field equations do not contain derivatives of order strictly higher than 2.

* By the way, why are these scalar fields called Galileons?: because the whole Lagrangian enjoys a "Galilean" symmetry

$$\pi \longrightarrow \pi + b_\mu x^\mu + c$$

* Let's analyze a bit more the conditions

1. Lagrangian is degenerate but, still, contains derivatives of π of order 2 or less. (This is not completely necessary in order to get purely second-order field equations (Deffayet et. al., Phys. Rev. D 2010).)

2. field equations are polynomial in second-order derivatives of π .

3. field equations do not contain undifferentiated or only once differentiated π .

4. field equations do not contain derivatives of order strictly higher than 2. (The Ostrogradsky's instability can be avoided even in the presence of higher-order field equations if the non-degeneracy condition is violated (Gleyzes et. al., Phys. Rev. Lett. 2015, Langlois et. al., JCAP 2016)).

* What's the Lagrangian for a single Galileon in D dimensions?

$$\mathcal{L}_N^{Gal,1} = (\mathcal{A}_{(2n+2)}^{\mu_1\dots\mu_{n+1}\nu_1\dots\nu_{n+1}}\pi_{\mu_{n+1}}\pi_{\nu_{n+1}})\pi_{\mu_1\nu_1}\dots\pi_{\mu_n\nu_n},$$

where

$$\mathcal{A}_{(2m)}^{\mu_1\mu_2\dots\mu_m\nu_1\nu_2\dots\nu_m} \equiv \frac{1}{(D-m)!}\epsilon^{\mu_1\mu_2\dots\mu_m\sigma_1\sigma_2\dots\sigma_{D-m}}\epsilon^{\nu_1\nu_2\dots\nu_m}{}_{\sigma_1\sigma_2\dots\sigma_{D-m}}$$

and $\pi_\mu \equiv \partial_\mu\pi, \quad \pi_{\mu\nu} \equiv \partial_\mu\partial_\nu\pi.$

* N indicates the number of times of π appearances:

$$N \equiv n+2\ (\geq 2), \quad N \leq D+1$$

* **Explicitly, if we are considering 4 dimensions, the Galileon Lagrangian contains four pieces:**

$$
\begin{aligned}
\mathcal{L}_2^{Gal,1} &= -\pi^\mu \pi_\mu \\
\mathcal{L}_3^{Gal,1} &= \pi^\mu \pi^\nu \pi_{\mu\nu} - \pi^\mu \pi_\mu \Box\pi \\
\mathcal{L}_4^{Gal,1} &= -(\Box\pi)^2(\pi_\mu\pi^\mu) + 2(\Box\pi)(\pi_\mu\pi^{\mu\nu}\pi_\nu) \\
&\quad +(\pi_{\mu\nu}\pi^{\mu\nu})(\pi_\rho\pi^\rho) - 2(\pi_\mu\pi^{\mu\nu}\pi_{\nu\rho}\pi^\rho) \\
\mathcal{L}_5^{Gal,1} &= -(\Box\pi)^3(\pi_\mu\pi^\mu) + 3(\Box\pi)^2(\pi_\mu\pi^{\mu\nu}\pi_\nu) \\
&\quad +3(\Box\pi)(\pi_{\mu\nu}\pi^{\mu\nu})(\pi_\rho\pi^\rho) \\
&\quad -6(\Box\pi)(\pi_\mu\pi^{\mu\nu}\pi_{\nu\rho}\pi^\rho) - 2(\pi_\mu{}^\nu\pi_\nu{}^\rho\pi_\rho{}^\mu)(\pi_\lambda\pi^\lambda) \\
&\quad -3(\pi_{\mu\nu}\pi^{\mu\nu})(\pi_\rho\pi^{\rho\lambda}\pi_\lambda) + 6(\pi_\mu\pi^{\mu\nu}\pi_{\nu\rho}\pi^{\rho\lambda}\pi_\lambda)
\end{aligned}
$$

* By partial integrations, we can get an equivalent Lagrangian:

$$\mathcal{L}_N^{Gal,3} = X \mathcal{A}_{(2n)}^{\mu_1 \ldots \mu_n \nu_1 \ldots \nu_n} \pi_{\mu_1 \nu_1} \ldots \pi_{\mu_n \nu_n} \quad ,$$

where $X \equiv \pi_\mu \pi^\mu$.

* By means of this Lagrangian, we can built the "generalized Galileons".

* The generalized Galileons are those scalar π fields in flat spacetime that have field equations containing derivatives or order 2 or less.

- Its construction is very easy (Deffayet et. al., Phys. Rev. D 2011). We just have to multiply the previous Lagrangian by arbitrary functions of π and X.

- The whole Lagrangian is the following:

$$\mathcal{L} = \sum_{n=0}^{D-1} \tilde{\mathcal{L}}_n\{f_n\},$$

where
$$\tilde{\mathcal{L}}_n\{f_n\} = f_n(\pi, X)\mathcal{L}_{N=n+2}^{Gal,3}$$
$$= f_n(\pi, X)(X\mathcal{A}_{(2n)}^{\mu_1\ldots\mu_n\nu_1\ldots\nu_n}\pi_{\mu_1\nu_1}\ldots\pi_{\mu_n\nu_n})$$

* Explicitly, if we are considering 4 dimensions, the generalized Galileon Lagrangian contains four pieces:

$$\tilde{\mathcal{L}}_0\{f_0\} = -f_0(\pi, X)X$$

$$\tilde{\mathcal{L}}_1\{f_1\} = -f_1(\pi, X)X\Box\pi$$

$$\tilde{\mathcal{L}}_2\{f_2\} = -f_2(\pi, X)X((\Box\pi)^2 - (\pi_{\mu\nu}\pi^{\mu\nu}))$$

$$\tilde{\mathcal{L}}_3\{f_3\} = -f_3(\pi, X)X((\Box\pi)^3 - 3(\Box\pi)(\pi_{\mu\nu}\pi^{\mu\nu}) + 2(\pi_\mu{}^\nu\pi_\nu{}^\rho\pi_\rho{}^\mu))$$

* Finally, we can covariantize the previous Lagrangian, so that we obtain the most general Lagrangian involving a scalar field and gravity, containing second-order derivatives or less of the scalar field and the metric, that leads to field equations of second order or less (Deffayet et. al., Phys. Rev. D 2009 and 2011).

* Replacing standard derivatives by covariant ones leads to higher-order field equations.

* Therefore, some counterterms will be needed.

* It is important to guarantee that the tensor sector contains the correct number of propagating degrees of freedom, i.e. 2.

* The Lagrangian in four dimensions is found to be (the first term in the Lagrangian can be found in (Woodard, Lect. Notes Phys. 2007))

$$\mathcal{L} = f(R) + G_{\mu\nu}\pi^{\mu}\pi^{\nu} + \sum_{N=2}^{5} \mathcal{L}_N^{Cov} \quad ,$$

where

$$\mathcal{L}_2^{Cov} = G_2(\pi, X)$$

$$\mathcal{L}_3^{Cov} = G_3(\pi, X)\Box\pi$$

$$\mathcal{L}_4^{Cov} = G_4(\pi, X)R + G_{4,X}\left((\Box\pi)^2 - (\pi_{\mu\nu}\pi^{\mu\nu})\right)$$

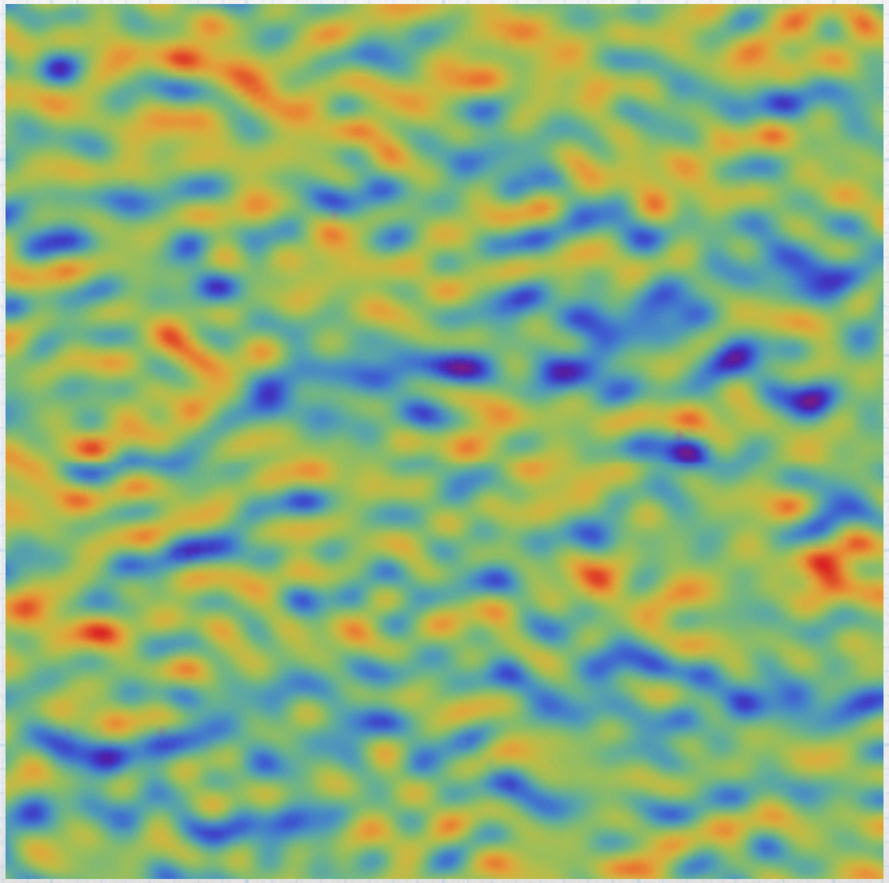
$$\mathcal{L}_5^{Cov} = G_5(\pi, X)G_{\mu\nu}\pi^{\mu\nu} - \frac{1}{6}G_{5,X}\left((\Box\pi)^3 - 3(\Box\pi)(\pi_{\mu\nu}\pi^{\mu\nu}) + 2({\pi_{\mu}}^{\nu}{\pi_{\nu}}^{\rho}{\pi_{\rho}}^{\mu})\right)$$

with $G_{N,X} = \dfrac{\partial G_N}{\partial X}$.

* It was shown by Kobayashi et. al. (Prog. Theor. Phys. 2011) that the previous construction is equivalent to the Horndeski's Lagrangian.
* If we want to build cosmological models that, in order to model inflation, dark energy, etc., make use of a scalar field, we must consider a generalized and covariantized Galileon (or the theories beyond that). This will avoid the Ostrogradsky's instability!!
* Effectively, many of the, well behaved, inflationary models presented in the literature incorporate Galileon fields.

Vector Galileons (no gauge symmetries)

* In cosmology, vector fields are also possible: because of their inherent privileged directions, they can generate anisotropy in the expansion and in the statistical distribution of fluctuations (stripes in the CMB map) (Dimopoulos et. al., JCAP 2009).



* Horndeski, in 1976 (J. Math. Phys. 1976), considered an Abelian vector field, with an action including sources, and with the assumption of recovering the Maxwell's equations in flat spacetime.
* Deffayet et. al. (Phys. Rev. D 2010) didn't invoke gauge invariances but, instead, studied several vector fields whose field equations are purely second-order.
* Fleury et. al. (JCAP 2014) coupled an Abelian vector field with a scalar field in the framework of Einstein's gravity.
* Deffayet et. al. (JHEP 2014) found a no-go theorem for an Abelian vector field in flat spacetime whose field equation is purely second-order.
* Heisenberg (JCAP 2014) studied a vector field, in curved spacetime, without gauge invariances.
* Heisenberg's work turned out not to be complete. My work (Allys et. al., JCAP 2016a and JCAP 2016b) completes Heisenberg's work (see also Heisenberg et. al., Phys. Lett. B 2016).
* Heisenberg et. al. (Phys. Lett. B 2016) and Kimura et. al. (2016) built the vector-tensor theories where the non degeneracy condition is violated.
* Allys et. al. (Phys. Rev. D 2016) built the vector-tensor theories with a non-Abelian SU(2) global gauge symmetry.

* Helmholtz decomposition tells us that

$$A_\mu = \partial_\mu \pi + \bar{A}_\mu$$

where $\partial^\mu \bar{A}_\mu = 0$.

* The longitudinal mode of the vector field is the scalar field π.

* If A_μ is a vector Galileon, then π is a scalar Galileon.

* Let's start in flat four-dimensional spacetime.

* We want to construct a Lagrangian for a vector field A_μ that contains derivatives of π of order 2 or less.

* That implies immediately that the Lagrangian must contain derivatives of A_μ of order 1 or less. The field equations for A_μ are, therefore, second order.

* There must be only three propagating degrees of freedom for the vector field.

* Procedure of investigation

1. We list all the possible terms which can be written as contractions of $(n-2)$ first-order derivatives of A_μ .
2. These test Lagrangians are linearly combined to provide the most general term at a given order n .
3. The Hessian, for each test Lagrangian, is computed. The necessary requirement $\mathcal{H}^{0\mu}=0$ for all $\mu=0,\cdots,3$ is used to derive relations among the coefficients of the linear combination, and to finally obtain the relevant terms that can give rise only three propagating degrees of freedom.
4. All the redundant Lagrangias (equivalent to others up to a total derivative) are removed.
5. Any term leading to a non-trivial dynamics for the scalar part that would be nonvanishing should be then set to zero in order to comply with the requirement that the scalar action is that provided by the Galileon.

* I will just show the procedure to find $\mathcal{L}_6$.

1. Test Lagrangians:

$$\mathcal{L}_{6,1}^{\text{test}} = (\partial \cdot A)^4 \qquad \mathcal{L}_{6,2}^{\text{test}} = (\partial \cdot A)^2 (\partial_\sigma A_\rho \partial^\sigma A^\rho)$$

$$\mathcal{L}_{6,3}^{\text{test}} = (\partial \cdot A)^2 (\partial_\rho A_\sigma \partial^\sigma A^\rho) \qquad \mathcal{L}_{6,4}^{\text{test}} = (\partial \cdot A) (\partial_\nu A_\sigma \partial^\rho A^\nu \partial^\sigma A_\rho)$$

$$\mathcal{L}_{6,5}^{\text{test}} = (\partial \cdot A) (\partial^\rho A^\nu \partial_\sigma A_\rho \partial^\sigma A_\nu) \qquad \mathcal{L}_{6,6}^{\text{test}} = (\partial_\mu A_\sigma \partial^\nu A^\mu \partial^\rho A_\nu \partial^\sigma A_\rho)$$

$$\mathcal{L}_{6,7}^{\text{test}} = (\partial^\nu A^\mu \partial_\rho A_\sigma \partial^\rho A_\mu \partial^\sigma A_\nu) \qquad \mathcal{L}_{6,8}^{\text{test}} = (\partial_\nu A^\sigma \partial^\nu A^\mu \partial_\rho A_\sigma \partial^\rho A_\mu)$$

$$\mathcal{L}_{6,9}^{\text{test}} = (\partial^\nu A^\mu \partial^\rho A_\mu \partial_\sigma A_\rho \partial^\sigma A_\nu) \qquad \mathcal{L}_{6,10}^{\text{test}} = (\partial_\nu A_\mu \partial^\nu A^\mu)^2$$

$$\mathcal{L}_{6,11}^{\text{test}} = (\partial_\mu A_\nu \partial^\nu A^\mu) (\partial_\sigma A_\rho \partial^\sigma A^\rho) \qquad \mathcal{L}_{6,12}^{\text{test}} = (\partial_\mu A_\nu \partial^\nu A^\mu) (\partial_\rho A_\sigma \partial^\sigma A^\rho)$$

2. Linear combination of test Lagrangians

$$\mathcal{L}_6 = \sum_{k=1}^{12} x_k \, \mathcal{L}_{6,k}^{\rm test}\,,$$

3. Hessian computation and imposition of the requirement $\mathcal{H}^{0\mu} = 0$

$$\mathcal{H}_6^{\mu\nu} = \frac{\partial^2 \mathcal{L}_6}{\partial(\partial_0 A_\mu)\partial(\partial_0 A_\nu)}\,,$$

This leads to

$$\mathcal{L}_6^{\rm Gal} = (\partial\cdot A)^4 - 2\,(\partial\cdot A)^2\,[(\partial_\rho A_\sigma \partial^\sigma A^\rho) + 2\,(\partial_\sigma A_\rho \partial^\sigma A^\rho)] + 8\,(\partial\cdot A)\,(\partial^\rho A^\nu \partial_\sigma A_\rho \partial^\sigma A_\nu) - (\partial_\mu A_\nu \partial^\nu A^\mu)^2$$
$$+4\,(\partial_\nu A_\mu \partial^\nu A^\mu)\,(\partial_\rho A_\sigma \partial^\sigma A^\rho) - 2\,(\partial_\nu A^\sigma \partial^\nu A^\mu \partial_\rho A_\sigma \partial^\rho A_\mu) - 4\,(\partial^\nu A^\mu \partial^\rho A_\mu \partial_\sigma A_\rho \partial^\sigma A_\nu)$$

$$\mathcal{L}_6^{\rm Perm} = (\partial\cdot A)^2\, F^{\mu\nu}F_{\mu\nu} - (\partial_\rho A_\sigma \partial^\sigma A^\rho)\, F^{\mu\nu}F_{\mu\nu} + 4\,(\partial\cdot A)\,\partial^\rho A^\nu \partial^\sigma A_\rho F_{\nu\sigma}$$
$$+\partial^\mu A_\nu F^\nu{}_\rho F^\rho{}_\sigma F^\sigma{}_\mu - 4\,\partial^\mu A_\nu \partial^\nu A_\rho \partial^\rho A_\sigma F^\sigma{}_\mu$$

$$\mathcal{L}_{FF\cdot FF} = (F_{\mu\nu}F^{\mu\nu})^2$$

$$\mathcal{L}_{FFFF} = F^\mu{}_\nu F^\nu{}_\rho F^\rho{}_\sigma F^\sigma{}_\mu$$

4. Setting to zero any term that leads to non-trivial dynamics for the scalar field.

$\mathcal{L}_6^{\mathrm{Gal}}$ leads to a higher than second-order field equation for π; therefore, it must be set to zero.

The other terms vanish when going to the scalar sector.

* The final Lagrangian in flat spacetime is the following:

$$\mathcal{L}_{\rm gen}(A_\mu) = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \sum_{n=2}^{6}\mathcal{L}_n \,,$$

with

$$\mathcal{L}_2 = f_2(A_\mu, F_{\mu\nu}, \tilde{F}_{\mu\nu}),$$

$$\mathcal{L}_3 = f_3\,(X)\,S_\mu{}^\mu \,,$$

$$\mathcal{L}_4 = f_4\,(X)\left[(S_\mu{}^\mu)^2 - S_\rho{}^\sigma S_\sigma{}^\rho\right] + g_4(X)A^\mu A_\lambda \tilde{F}_{\mu\nu}S^{\lambda\nu} \,,$$

$$\mathcal{L}_5 = f_5\,(X)\left[(S_\mu{}^\mu)^3 - 3(S_\mu{}^\mu)S_\rho{}^\sigma S_\sigma{}^\rho + 2S_\rho{}^\sigma S_\sigma{}^\gamma S_\gamma{}^\rho\right] + g_5\,(X)\,\tilde{F}^{\alpha\mu}\tilde{F}^\beta{}_\mu S_{\alpha\beta} \,,$$

$$\mathcal{L}_6 = g_6\,(X)\,\tilde{F}^{\alpha\beta}\tilde{F}^{\mu\nu}S_{\alpha\mu}S_{\beta\nu},$$

where $X \equiv A_\mu A^\mu \,,$ and $S^{\mu\nu} \equiv \partial^\mu A^\nu + \partial^\nu A^\mu \,.$

* Let's go now to curved four-dimensional spacetime.
* The procedure of investigation is exactly equal to the described before but we have to take into account the following aspects:

1. The standard derivatives must be replaced by covariant ones.
2. This can lead to higher than second-order field equations. Therefore, some counterterms must be added.
3. We have to include couplings between the vector field or the field strength tensor with the curvature that vanish when going to flat spacetime.

* **The final Lagrangian in curved spacetime is the following:**

$$\mathcal{L}_{\text{gen}}(A_\mu) = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \mathcal{L}^{\text{Curv}} + \sum_{n=2}^{6}\mathcal{L}_n \,,$$

with

$$\mathcal{L}^{\text{Curv}} = f_1^{\text{Curv}} G_{\mu\nu}A^\mu A^\nu + f_2^{\text{Curv}}(X) L_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma},$$

$$\mathcal{L}_2 = f_2(A_\mu, F_{\mu\nu}, \tilde{F}_{\mu\nu}),$$

$$\mathcal{L}_3 = f_3\,(X)\,S_\mu{}^\mu \,,$$

$$\mathcal{L}_4 = f_4(X)R - 2f_{4,X}\,(X)\left[(S_\mu{}^\mu)^2 - S_\rho{}^\sigma S_\sigma{}^\rho\right] + g_4(X)A^\mu A_\lambda \tilde{F}_{\mu\nu}S^{\lambda\nu} \,,$$

$$\mathcal{L}_5 = f_5(X)G_{\mu\nu}S^{\mu\nu} + 6f_{5,X}\,(X)\left[(S_\mu{}^\mu)^3 - 3(S_\mu{}^\mu)S_\rho{}^\sigma S_\sigma{}^\rho + 2S_\rho{}^\sigma S_\sigma{}^\gamma S_\gamma{}^\rho\right]$$
$$+g_5(X)\tilde{F}^{\alpha\mu}\tilde{F}^\beta{}_\mu S_{\alpha\beta} \,,$$

$$\mathcal{L}_6 = g_6\,(X)\,\tilde{F}^{\alpha\beta}\tilde{F}^{\mu\nu}S_{\alpha\mu}S_{\beta\nu},$$

* The four-rank divergence-free tensor $L_{\mu\nu\rho\sigma}$ is

$$L_{\mu\nu\rho\sigma} = 2R_{\mu\nu\rho\sigma} + 2(R_{\mu\sigma}g_{\rho\nu} + R_{\rho\nu}g_{\mu\sigma} - R_{\mu\rho}g_{\nu\sigma} - R_{\nu\sigma}g_{\mu\rho}) + R(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\rho\nu}).$$

* The use of divergence-free tensors permits us to avoid higher-order derivatives of the metric in the equations of motion.

* One interesting aspect of the whole Lagrangian is the appearance of parity-violating terms (those that involve odd copies of $\tilde{F}_{\mu\nu}$). They can lead to observational signatures in the CMB.

* If we want to build cosmological models that, in order to model inflation, dark energy, etc., make use of a vector field, we must consider a generalized and covariantized vector Galileon (or the theories beyond that). This will avoid the Ostrogradsky's instability!!

* However, we still need to consider (with or without gauge invariances):

* Couplings with a scalar field (e.g., the fF^2 model (Watanabe et. al., Phys. Rev. Lett. 2009)).

* Or a rapidly oscillating vector field (e.g. the vector curvaton scenario (Dimopoulos, Phys. Rev. D 2006)).

* Or the multi-vector field case (e.g. the gauge-flation model (Maleknejad et. al., Phys. Lett. B 2013)).

This is to avoid the highly anisotropization that produces just one vector field.

Non-Abelian vector Galileons

* We are working in the framework of special unitary global gauge transformations that, of course, are part of a simple Lie group.

* A gauge vector field then transforms as

$$\delta A^a_\mu = i\epsilon^b (T_b^A)^a{}_c A^c_\mu,$$

where ϵ^b represents the amount of the transformation and T^A represents the matrices that conform the adjoint representation of the Lie group:

$$(T_c^A)^a{}_b = -i f^a{}_{bc}.$$

* The $f^a{}_{bc}$ are the structure constants of the Lie group:

$$[T_a, T_b] = i f_{abc} T^c.$$

* The T_a are the generators of the gauge transformations.

* We are working with global gauge transformations since the gauge field transforms in this case according to the adjoint representation of the Lie group.

* Following the same strategy as in the Abelian case, and considering only up to six space-time indices, we have obtained the following Lagrangian:

$$\mathcal{L}_2 = \tilde{f}(A_\mu^a, F_{\mu\nu}^a, \tilde{F}_{\mu\nu}^a).$$

$$\begin{aligned}
\mathcal{L}_4^1 &= (A_b \cdot A^b)\left[(\nabla \cdot A_a)(\nabla \cdot A^a) - (\nabla_\mu A_a^\nu)(\nabla^\mu A_\nu^a) + \frac{1}{4} A_a \cdot A^a R\right] \\
&\quad + 2(A_a \cdot A_b)\left[(\nabla \cdot A^a)\left(\nabla \cdot A^b\right) - (\nabla_\mu A^{\nu a})(\nabla^\mu A_\nu^b) + \frac{1}{2} A^a \cdot A^b R\right], \\
\mathcal{L}_4^2 &= (A_a \cdot A_b)\left[(\nabla \cdot A^a)\left(\nabla \cdot A^b\right) - (\nabla_\mu A^{\nu a})(\nabla^\mu A_\nu^b) + \frac{1}{4} A^a \cdot A^b R\right] \\
&\quad + (A^{\mu a} A^{\nu b})\left[(\nabla_\mu A_a^\alpha)(\nabla_\nu A_{\alpha b}) - (\nabla_\nu A_a^\alpha)(\nabla_\mu A_{\alpha b}) - \frac{1}{2} A_b^\rho A^{\sigma b} R_{\mu\nu\rho\sigma}\right], \\
\mathcal{L}_4^3 &= \tilde{G}_{\mu\sigma}^b A_a^\mu A_{\alpha b} S^{\alpha\sigma a},
\end{aligned}$$

where $G_{\mu\nu}^a \equiv \partial_\mu A_\nu^a - \partial_\nu A_\mu^a$ and $S_{\mu\nu}^a \equiv \partial_\mu A_\nu^a + \partial_\nu A_\mu^a$.

* We have in addition

$$\mathcal{L}_1^{\text{curv}} = G_{\mu\nu} A^{\mu a} A^{\nu}_{a},$$

$$\mathcal{L}_2^{\text{curv}} = L_{\mu\nu\rho\sigma} F^{\mu\nu}_{a} F^{a}_{\mu\nu},$$

$$\mathcal{L}_3^{\text{curv}} = L_{\mu\nu\rho\sigma} \epsilon_{abc} F^{\mu\nu a} A^{\rho b} A^{\sigma c},$$

$$\mathcal{L}_4^{\text{curv}} = L_{\mu\nu\rho\sigma} A^{\mu a} A^{\nu b} A^{\rho}_{a} A^{\sigma}_{b}.$$

* Finally, we will be in the very interesting position of finding out cosmologically viable models with either vanishing or very tiny levels of anisotropy, in agreement with observational data (e.g., the gauge-flation model (Maleknejad et. al., Phys. Lett. B 2013)).

* We are currently working in the cosmological consequences of the terms $G_{\mu\nu} A^{\mu a} A_a^{\nu}$, and $L_{\mu\nu\rho\sigma} A^{\mu a} A^{\nu b} A_a^{\rho} A_b^{\sigma}$ (Navarro et. al., work in progress).

* At the fundamental level, we will be approaching more to the challenge of merging cosmology and particle physics.

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