Geometrothermodynamics of black holes

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- Motivation
- Fundamentals of GTD (ideal gas)
  Thermodynamic phase space
  Space of equilibrium states
- Examples: Kerr black hole
- Conclusions

- **... ONE METRIC ... $\to$ ONE CONNECTION**
  - Theory of relativity (Einstein, 1915):

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \kappa T_{\mu\nu}$$

curvature = gravitational interaction

Symmetry $\longrightarrow$ Diffeomorphism invariance

- **... ONE METRIC ... $\to$ SEVERAL CONNECTIONS**

— Gauge field theories (Yang-Mills, 1953):

$$\text{Minkowski metric} \begin{array}{c} \nearrow \\ \to \\ \searrow \end{array} \begin{array}{ccc} U(1) - \text{connection} & \to & \text{electromagnetism} \\ SU(2) - \text{connection} & \to & \text{weak interaction} \\ SU(3) - \text{connection} & \to & \text{strong interaction} \end{array}$$

curvature = field interaction

Symmetry $\longrightarrow$ Gauge invariance

- **WHAT ABOUT THERMODYNAMICS?**

... curvature = thermodynamic interaction ?

– Several attempts: Gibbs (1912), Caratheodory (1925), Fisher-Rao (1945), Hermann (1963), Weinhold (1975), Ruppeiner (1979), ...

Hessian metrics on the equilibrium space ...

– Quevedo (2007) **"Geometrothermodynamics"**, JMP, **48**, 013506

Legendre invariant metrics on the phase space ...

Main postulate:

Symmetry $\longrightarrow$ Legendre invariance

EXAMPLE: THE IDEAL GAS ($n = 2$)

EQUATIONS OF STATE:

$$\boxed{PV = Nk_BT \quad U = \frac{3}{2}Nk_BT}$$

FUNDAMENTAL EQUATION ($N = const$)**:**

$$\boxed{S = S(U,V) = \frac{3}{2}Nk_B \ln U + Nk_B \ln V}$$

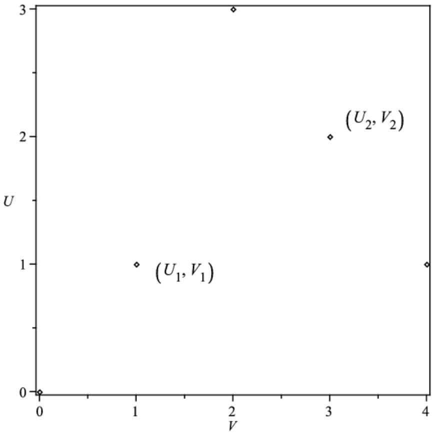
FIRST LAW OF THERMODYNAMICS (FLT):

$$\boxed{dU = TdS - PdV} \quad \leftrightarrow \quad dS = \frac{1}{T}dU + \frac{P}{T}dV$$

FUNDAMENTAL EQUATION + FLT = EQUATIONS OF STATE :

$$S = S(U,V) \rightarrow dS = \frac{\partial S}{\partial U}dU + \frac{\partial S}{\partial V}dV \rightarrow \frac{\partial S}{\partial U} = \frac{1}{T} \quad \frac{\partial S}{\partial V} = \frac{P}{T}$$

EQUILIBRIUM SPACE OF THE IDEAL GAS ($n = 2$)



FUNDAMENTAL EQUATION ($N = const$)**:**

$$S = \frac{3}{2} N k_B \ln U + N k_B \ln V \leftrightarrow U = \left(e^S V^{-1}\right)^{2/3}$$

THE IDEA OF EQUILIBRIUM GTD

GEOMETRY	$\Longleftrightarrow$	**THERMODYNAMICS**
Equilibrium space $\mathcal{E}$	$\Longleftrightarrow$	**Thermodynamic system**
Curvature of $\mathcal{E}$	$\Longleftrightarrow$	**Thermodynamic interaction**
Singularity of $\mathcal{E}$	$\Longleftrightarrow$	**Phase transition**
TD geodesic of $\mathcal{E}$	$\Longleftrightarrow$	**Free quasi-static process**
Diff. invariance of $\mathcal{E}$	$\Longleftarrow$	**Representation independent**
Legendre invariance of $\mathcal{T}$	$\Longleftrightarrow$	**TD potential independent**

EXAMPLE: THE IDEAL GAS ($n = 2$)

- **FUNDAMENTAL EQUATION**

$\Phi = U$, $E^a = \{S, V\}$, $I^a = \{T, P\}$, $\mathrm{Ex}: U = \left(e^S V^{-1}\right)^{2/3}$

$$\boxed{\Phi = \Phi(E^a) \quad a = 1, 2, ..., n}$$

- **FIRST LAW OF THERMODYNAMICS**

$dU = TdS - PdV \rightarrow dS = \frac{1}{T}dU + \frac{P}{T}dV$

$$\boxed{d\Phi = I_a dE^a}$$

- **LEGENDRE TRANSFORMATIONS**

$F = U - TS$	partial Legendre transformation
$H = U + PV$	partial Legendre transformation
$G = U - TS + PV$	total Legendre transformation

EQUILIBRIUM SPACE

Differential manifold	$\rightarrow$	$\mathcal{E}$
Coordinates of $\mathcal{E}$	$\rightarrow$	$E^a \quad a = 1, ..., n$
Conditions on $\mathcal{E}$	$\rightarrow$	$\Phi = \Phi(E^a), \; d\Phi = I_a dE^a$
Metric on $\mathcal{E}$	$\rightarrow$	$g = ds^2_{\mathcal{E}} = g_{ab} dE^a dE^b$

LEGENDRE TRANSFORMATIONS IN GENERAL

Let the coordinates $\{Z^A\} = \{\Phi, E^a, I^a\}$ be independent:

$$\{Z^A\} = \{\Phi, E^a, I^a\} \longrightarrow \{\widetilde{Z}^A\} = \{\tilde{\Phi}, \tilde{E}^a, \tilde{I}^a\}$$

$$\Phi = \tilde{\Phi} - \delta_{kl}\tilde{E}^k \tilde{I}^l \ , \quad E^i = -\tilde{I}^i, \ \ E^j = \tilde{E}^j, \quad I^i = \tilde{E}^i, \ \ I^j = \tilde{I}^j$$

where $i \cup j$ disjoint decomposition of $\{1, ..., n\}$, and $k, l = 1, ..., i$.

Total Legendre transformation if $i = \{1, ..., n\}$.
Identity transformation $i = \emptyset$.

Legendre invariance: $I_a = \delta_{ab} I^b$

$$\Theta = d\Phi - I_a dE^a \rightarrow \tilde{\Theta} = d\tilde{\Phi} - \tilde{I}_a d\tilde{E}^a$$

THERMODYNAMIC PHASE SPACE

Phase space = Riemannian contact manifold $(\mathcal{T}, \Theta, G)$

Differential manifold	$\rightarrow$	$\mathcal{T}$
Coordinates of $\mathcal{T}$	$\rightarrow$	$Z^A = \{\Phi, E^a, I^a\}$
Contact 1-form	$\rightarrow$	$\Theta = d\Phi - I_a dE^a$
Metric on $\mathcal{T}$	$\rightarrow$	$G = ds^2_{\mathcal{T}} = G_{AB} dZ^A dZ^B$

Legendre invariance:

$$\Theta = d\Phi - I_a dE^a \rightarrow \tilde{\Theta} = d\tilde{\Phi} - \tilde{I}_a d\tilde{E}^a$$

$$G = G_{AB} dZ^A dZ^B \rightarrow \tilde{G} = \tilde{G}_{AB} d\tilde{Z}^A d\tilde{Z}^B$$

GEOMETRIC STRUCTURE OF GTD

Equilibrium space **Phase space**

$$
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{\varphi} & \mathcal{T} \\
\downarrow & & \downarrow \\
T_{\mathcal{E}} & \xrightarrow{\varphi_*} & T_{\mathcal{T}} \\
\downarrow & & \downarrow \\
T^*_{\mathcal{E}} & \xleftarrow{\varphi^*} & T^*_{\mathcal{T}}
\end{array}
$$

$$\boxed{\varphi^*(\Theta) = 0 \qquad \varphi^*(G) = g}$$

LEGENDRE INVARIANT METRIC

PHASE SPACE:

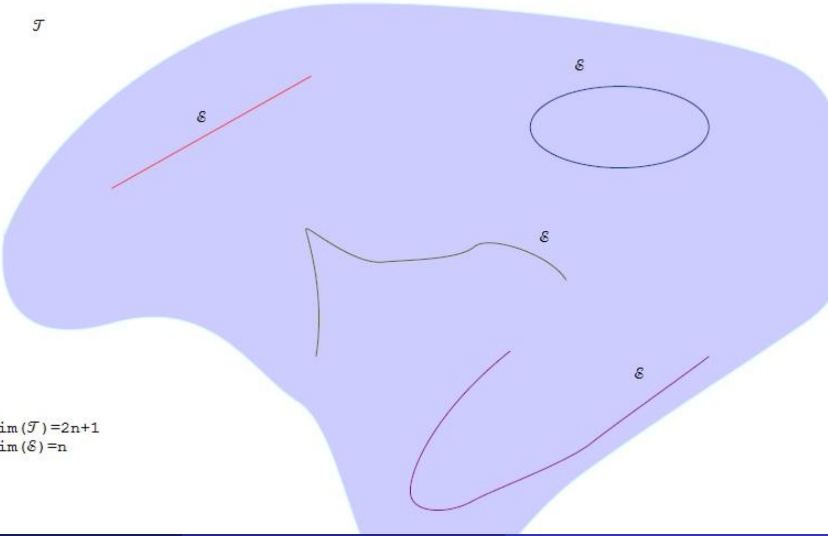
$$G^{II} = (d\Phi - I_a dE^a)^2 + \Lambda\,(\chi_{ab} E^a I^b)(\eta_{ab} dE^a dI^b)$$ (total inv.)

$$\Lambda = const. \qquad \chi_{ab} = \mathrm{diag}(c_1, c_2, ..., c_n)\ ,\ \eta_{ab} = \mathrm{diag}(-1, 1, ..., 1)$$

EQUILIBRIUM SPACE: $(\Lambda = 1)$

$$g = \Phi\ \left(\eta_a{}^b \frac{\partial^2 \Phi}{\partial E^b \partial E^c} dE^a dE^c \right)$$

GEOMETROTHERMODYNAMICS



dim($\mathcal{T}$)=2n+1
dim($\mathcal{E}$)=n

GTD of Kerr black hole

Fundamental equation $\Phi = S \ , \ E^a = (M, J)$

$$S = 2\pi \left(M^2 + \sqrt{M^4 - J^2} \right) \qquad r_\pm = M \pm \sqrt{M^2 - J^2/M^2}$$

Thermodynamic metric $g^K = S \left(S_{MM} dM^2 - S_{JJ} dJ^2 \right)$

$$g^K = \frac{16\pi^2 r_+^2 (r_+ + r_-)}{(r_+ - r_-)^4} \left[r_+ (r_+^2 - 6 r_+ r_- - 3 r_-^2) dM^2 + (r_+ + r_-) dJ^2 \right]$$

Thermodynamic curvature

$$R^K = \frac{(3r_+^3 + 3r_+^2 r_- + 17 r_+ r_-^2 + 9 r_-^3)(r_+ - r_-)^3}{2\pi^2 r_+^2 (r_+ + r_-)^4 (r_+^2 - 6 r_+ r_- - 3 r_-^2)^2}$$

Heat capacity

$$C_J == \frac{2\pi^2 r_+ (r_+ + r_-)^2 (r_+ - r_-)}{r_+^2 - 6 r_+ r_- - 3 r_-^2}$$

Curvature of $\mathcal{E}_{BH}$ = Thermodynamic interaction of BH's
Curvature singularities of $\mathcal{E}_{BH}$ = Phase transitions of BH's

COMPARISON

System	Weinhold (PhT)	Ruppeiner (PhT)	GTD (PhT)
RN	curved (yes)	flat (no)	curved (yes)
Kerr	flat (no)	curved (yes)	curved (yes)
BTZ	curved (yes)	flat (no)	curved (yes)
2Dgrav	curved (yes)	flat (no)	curved (yes)
HighD	?	curved/flat(?)	curved (yes)
EGB	flat (no)	curved(yes)	curved (yes)
Horava	curved (no)	curved (no)	curved (yes)
Born-Infeld	curved (no)	flat (no)	curved (yes)
Phanton-AdS	curved (no)	flat (no)	curved (yes)

Non-Legendre invariant metrics:

$$g^W = \frac{\partial^2 U}{\partial E^a \partial E^b} dE^a dE^b \qquad g^R = \frac{\partial^2 S}{\partial E^a \partial E^b} dE^a dE^b$$

Legendre invariant metric:

$$\boxed{g^{I/II} = \Phi \left(\eta_a{}^b \frac{\partial^2 \Phi}{\partial E^b \partial E^c} dE^a dE^c \right)}$$

OTHER APPLICATIONS

RELATIVISTIC COSMOLOGY

– A standard cosmological model with only one fundamental equation

CHEMISTRY

– A chemical reaction can be described as a thermodynamic geodesic

NON-EQUILIBRIUM THERMODYNAMICS

– Control manifolds as non-equilibrium models

EMERGENT/THERMODYNAMIC GRAVITY

– GR is a thermodynamically non-interacting system

RELATIVISTIC ASTROPHYSICS

— How physical are exact solutions of Einstein's equations?

ECONOPHYSICS

– Income distribution as Riemannian manifolds

OTHER APPLICATIONS (IN PROGRESS)

RELATIVISTIC GTD ???

$$T = T_0 \quad T = \sqrt{1 - v^2} T_0 \quad T = \frac{T_0}{\sqrt{1 - v^2}} ???$$

STATISTICAL GTD ???

– Statistical model for black holes ????

QUANTUM GTD ???

– Quantum black holes ???

GENERAL CONCLUSIONS

• GTD is a consistent, invariant formalism describing thermodynamics in terms of geometric concepts

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Diff. invariance of $\mathcal{E}$ **Legendre invariance of** $\mathcal{T}$	$\Longleftarrow$ $\Longleftrightarrow$	**Representation independent** **TD potential independent**