RELATIVISTIC STATIC THIN DISKS OF POLARIZED OR MAGNETIZED MATTER

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1 Introduction

The study of electromagnetic fields in relativistic astrophysics has proved to be of high relevance, however polarized matter as a source of the fields have been studied very little in the context of the exact solutions of the Einstein-Maxwell equations for physical objects, that is why we believe it is important to include that type of sources to the study of axially symmetric disk-like configurations of matter which are solutions of the Einsten-Maxwell field equations.

2 Field Equations And Thin Disk Solutions

An static axially symmetric spacetime with a thin disk in the surface $z = 0$,

$$g_{\alpha\beta}(r, z) = g_{\alpha\beta}(r, -z)\,, \tag{1}$$

$$g_{\alpha\beta,z}(r, z) = -g_{\alpha\beta,z}(r, -z)\,. \tag{2}$$

We assume that the metric tensor is continuous through the disk,

$$[g_{\alpha\beta}] = 0\,, \tag{3}$$

with a finite discontinuity in its first normal derivative,

$$b_{\alpha\beta} = [g_{\alpha\beta,z}] = 2g_{\alpha\beta,z}|_{z=0^+}\,,$$

where the reflection symmetry of the metric tensor has been used.

We assume an electromagnetic four potential A_α with reflection symmetry with respect to the surface $z = 0$,

$$A_\alpha(r, z) = A_\alpha(r, -z)\,, \tag{4}$$

$$A_{\alpha,z}(r, z) = -A_{\alpha,z}(r, -z)\,. \tag{5}$$

As we assumed with the metric tensor, the electromagnetic four potential is continuous through the disk,

$$[A_\alpha] = 0\,, \tag{6}$$

with a finite discontinuity in its first normal derivative, expressed as

$$[A_{\alpha,z}] = 2A_\alpha, z|_{z=0^+}\,, \tag{7}$$

where the reflection symmetry of the potential has been used.

The Einstein-Maxwell equations for a continuum media,

$$G_{\alpha\beta} = T^{M}_{\alpha\beta} + T^{F}_{\alpha\beta} + T^{FM}_{\alpha\beta} \ , \tag{8}$$

$$F^{\alpha\beta}{}_{;\beta} = M^{\alpha\beta}{}_{;\beta} \ , \tag{9}$$

where $F_{\alpha\beta}$ is the electromagnetic tensor,

$$F_{\alpha\beta} = A_{\beta,\alpha} - A_{\alpha,\beta} \tag{10}$$

and $M_{\alpha\beta}$ is the polarization-magnetization tensor, defined in terms of the electric polarization vector P_{α} and the magnetic polarization vector M_{α},

$$M^{\alpha\beta} = P^{\alpha}u^{\beta} - P^{\beta}u^{\alpha} + \epsilon^{\alpha\beta\mu\nu}M_{\mu}u_{\nu} \ , \tag{11}$$

where u_{α} is the 4-velocity vector of the observer and ϵ is the LeviCivita Tensor.

The energy-momentum tensor has a component due to the matter $T_{\alpha\beta}^{M}$, one due to the electromagnetic fields $T_{\alpha\beta}^{F}$ and one due to the electromagnetic interaction with the polarized matter $T_{\alpha\beta}^{FM}$,

$$T_{\alpha\beta} = T_{\alpha\beta}^{M} + T_{\alpha\beta}^{F} + T_{\alpha\beta}^{FM} \,, \tag{12}$$

where

$$T_{\alpha\beta}^{F} = F_{\alpha\mu} F_{\beta}{}^{\mu} - \frac{1}{4} g_{\alpha\beta} F_{\mu\nu} F^{\mu\nu} \,, \tag{13}$$

and

$$T_{\alpha\beta}^{FM} = -F_{\alpha\mu} M_{\beta}{}^{\mu} \,. \tag{14}$$

Now, we write the metric tensor as

$$g_{\alpha\beta} = g^{+}_{\alpha\beta}\Theta(z) + g^{-}_{\alpha\beta}\left\{1 - \Theta(z)\right\} \ , \tag{15}$$

with $\Theta(z)$ the Heaviside distribution, and the Einstein tensor as

$$G_{\alpha\beta} = G^{+}_{\alpha\beta}\Theta(z) + G^{-}_{\alpha\beta}\left\{1 - \Theta(z)\right\} + Q_{\alpha\beta}\delta(z) \, . \tag{16}$$

Accordingly,

$$G^{\pm}_{\alpha\beta} = R^{\pm}_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R^{\pm}, \tag{17}$$

$$Q_{\alpha\beta} = H_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}H \, , \tag{18}$$

$$H_{\alpha\beta} = \frac{1}{2}\left(b^{z}{}_{\alpha}\delta^{z}{}_{\beta} + b^{z}{}_{\beta}\delta^{z}{}_{\alpha} - b^{\mu}{}_{\mu}\delta^{z}{}_{\alpha}\delta^{z}{}_{\beta} - g^{zz}b_{\alpha\beta}\right) \ , \tag{19}$$

where $R_{\alpha\beta}$ is the Ricci tensor for the outside region, $H_{\alpha\beta}$ is the Ricci tensor in the disk, and $H = g^{\alpha\beta}H_{\alpha\beta}$.

Regarding that the matter distribution is only in $z = 0$,

$$T_M^{\alpha\beta} = \tau_M^{\alpha\beta}\delta(z)\,, \tag{20}$$

$$M^{\alpha\beta} = \Pi^{\alpha\beta}\delta(z)\,. \tag{21}$$

where $\tau_M^{\alpha\beta}$ stands for the energy-momentum tensor matter component in the disk and $\Pi^{\alpha\beta}$ for the polarization-magnetization tensor of the disk. In consequence, the electric and magnetic polarization vectors of the disk can be expressed as

$$P_\alpha = u^\beta \Pi_{\beta\alpha}\,, \tag{22}$$

$$M^\alpha = \frac{1}{2}\varepsilon^{\alpha\beta\mu\nu}\Pi_{\mu\nu}u_\beta\,. \tag{23}$$

The electromagnetic energy-momentum tensor

$$T_F^{\alpha\beta} = T_F^{\alpha\beta^+} \Theta(z) + T_F^{\alpha\beta^-} \left\{1 - \Theta(z)\right\} \,, \tag{24}$$

$$T_{FM}^{\alpha\beta} = \tau_{FM}^{\alpha\beta} \delta(z) \,, \tag{25}$$

where $\tau_{FM}^{\alpha\beta}$ stands for the electromagnetic interaction tensor in the disk, given by

$$\tau_{FM}^{\alpha\beta} = \bar{F}^{\alpha}_{\ \mu} \Pi^{\mu\beta} \,, \tag{26}$$

where $\bar{F}^{\alpha}_{\ \mu}$ is the average electromagnetic tensor through the disk,

$$\bar{F}^{\alpha}_{\ \mu} = \frac{F^{\alpha\ +}_{\ \mu} + F^{\alpha\ -}_{\ \mu}}{2} \,. \tag{27}$$

The covariant derivatives of the electromagnetic tensor and the polarization-magnetization tensor can be expressed as

$$\sqrt{-g}F^{\alpha\beta}{}_{;\beta} = \left(\hat{F}^{\alpha\beta}{}_{,\beta}\right)^{D} + \left[\hat{F}^{\alpha\beta}\right]\delta^{z}{}_{\beta}\delta(z), \tag{28}$$

$$\sqrt{-g}M^{\alpha\beta}{}_{;\beta} = \hat{\Pi}^{\alpha\beta}{}_{,\beta}\delta(z) + \hat{\Pi}^{\alpha z}\delta'(z)\,, \tag{29}$$

with

$$\hat{F}^{\alpha\beta} = \sqrt{-g}F^{\alpha\beta}\,, \tag{30}$$

$$\hat{\Pi}^{\alpha\beta} = \sqrt{-g}\Pi^{\alpha\beta}\,. \tag{31}$$

Then, we substitute the equations (20), (25) and (24) in the equations (8) and (9), and obtain the electrovacuum field equations

$$G^{\pm}_{\alpha\beta} = \left(T^{F}_{\alpha\beta}\right)^{\pm} \,, \tag{32}$$

$$\hat{F}^{\alpha\beta}_{\pm,\beta} = 0 \,, \tag{33}$$

for $z \geq 0$ and $z \leq 0$, and

$$Q_{\alpha\beta} = \tau^{M}_{\alpha\beta} + \tau^{FM}_{\alpha\beta} \,, \tag{34}$$

$$\left[\hat{F}^{\alpha z}\right] = \hat{\Pi}^{\alpha\beta}{}_{,\beta} \,, \tag{35}$$

$$\hat{\Pi}^{\alpha z} = 0 \,, \tag{36}$$

for $z = 0$, which are the field equations of the disk.

From (34) we obtain the matter energy-momentum tensor

$$T^M_{\alpha\beta} = \left(Q_{\alpha\beta} - \tau^{FM}_{\alpha\beta}\right)\delta(z)\,, \tag{37}$$

and the surface energy-momentum tensor of the disk

$$S_{\alpha\beta} = \int T^M_{\alpha\beta}\mathrm{d}s_n = \sqrt{g_{zz}}\left(Q_{\alpha\beta} - \tau^{FM}_{\alpha\beta}\right)\,, \tag{38}$$

where $ds_n = \sqrt{g_{zz}}dz$ is the normal length to the disk.

Now, in the orthonormal tetrad

$$e^{\alpha}_{(t)} = e^{-\psi}\delta^{\alpha}_{t} \, , \tag{39}$$

$$e^{\alpha}_{(r)} = e^{\psi}\delta^{\alpha}_{r} \, , \tag{40}$$

$$e^{\alpha}_{(\phi)} = e^{\psi}\delta^{\alpha}_{\phi}/r \, , \tag{41}$$

$$e^{\alpha}_{(z)} = e^{\psi}\delta^{\alpha}_{z} \, , \tag{42}$$

the surface energy-momentum tensor can be written as

$$S_{\alpha\beta} = \sigma e^{(t)}_{\alpha} e^{(t)}_{\beta} + p_r e^{(r)}_{\alpha} e^{(r)}_{\beta} + p_\phi e^{(\phi)}_{\alpha} e^{(\phi)}_{\beta} + p_z e^{(z)}_{\alpha} e^{(z)}_{\beta} \, , \tag{43}$$

where σ is the surface energy density, p_r is the radial pressure, p_ϕ is the azimuthal pressure and p_z is the normal pressure of the disk.

3 Electrically Polarized Solution

In a conformastatic spacetime,

$$ds^2 = -e^{2\psi}dt^2 + e^{-2\psi}\left(dr^2 + r^2 d\phi^2 + dz^2\right) \ , \tag{44}$$

where the metric function ψ is dependent on the coordinates (r, z). For the electromagnetic four potential we take

$$A_\alpha = (-\chi, 0, 0, 0) \ . \tag{45}$$

with χ depending on the coordinates (r, z).

The only non-zero components of the polarization-magnetization tensor are

$$\Pi_{rt} = -\Pi_{tr} = P_r e^{\psi} \ . \tag{46}$$

Now, equations (32) and (33) yield the system of equations

$$\chi_{,r}^2 - 2e^{2\psi}\psi_{,r}^2 = 0\,, \tag{47}$$

$$\chi_{,z}^2 - 2e^{2\psi}\psi_{,z}^2 = 0\,, \tag{48}$$

$$\chi_{,r}\chi_{,z} - 2e^{2\psi}\psi_{,r}\psi_{,z} = 0\,, \tag{49}$$

$$\nabla^2\psi - \nabla\psi\cdot\nabla\psi = 0\,. \tag{50}$$

With equations (47), (48) and (49) we found

$$\chi = \sqrt{2}\left(e^{\psi} - 1\right) \, . \tag{51}$$

From equation (35) we obtain a differential equation for P_r

$$rP_{r,r} + P_r(1 - r\psi_{,r}) + 2\sqrt{2}r\psi_{,z} = 0\,, \tag{52}$$

whose solution is given by

$$P_r = -\frac{2\sqrt{2}e^{\psi}}{r}\int r\psi_{,z}e^{-\psi}\mathrm{d}r\,, \tag{53}$$

where we choose the integration constant equal to zero.

The nonzero components of the surface energy-momentum tensor in the orthonormal tetrad

$$\sigma = S_{(t)(t)} = e^{\psi} \left(4\psi_{,z} + \sqrt{2}\psi_{,r} P_r \right) \,, \tag{54}$$

$$p_r = S_{(r)(r)} = -\sqrt{2}\psi_{,r} e^{\psi} P_r \,, \tag{55}$$

and the radial component of the polarization vector,

$$P_{(r)} = -\frac{2\sqrt{2}e^{2\psi}}{r} \int r\psi_{,z} e^{-\psi} \mathrm{d}r \,. \tag{56}$$

On the other hand, the equation (50) can be equally expressed as

$$\nabla^2 \left(e^{-\psi}\right) = 0\,, \tag{57}$$

which is the Laplace equation $\nabla^2 U = 0$, therefore we can write the metric function as

$$e^{-\psi} = 1 - U\,, \tag{58}$$

with U a solution of Laplace equation dependent on the coordinates (r, z). Accordingly, equation (58) allows us to write the physical quantities of the disk in terms of the function U.

We have

$$\sigma = \frac{4}{(1-U)^2}\left[U_{,z} - \frac{U_{,r}}{(1-U)r}\int_0^\infty U_{,z} r\mathrm{d}r\right], \tag{59}$$

$$p_r = \frac{4U_{,r}}{r(1-U)^3}\int_0^\infty U_{,z} r\mathrm{d}r\,, \tag{60}$$

$$P_{(r)} = \frac{2\sqrt{2}}{(1-U)^2 r}\int_0^\infty U_{,z} r\mathrm{d}r\,. \tag{61}$$

It can be seen that, the physical features of the disk depend entirely on the election of the solution of Laplace equation U.

4 A Particular Family Of Polarized Disks

Considering the axial symmetry of the system, we take as solution of Laplace equation the function

$$U_n = -\sum_{l=0}^{n} \frac{C_l P_l\,(z/R)}{R^{l+1}}\,, \tag{62}$$

with C_l constants, $P_l(\cos\theta)$ the Legendre polynomials and

$$R = \sqrt{r^2 + z^2}\,, \tag{63}$$

this solution of the Laplace equation decays at infinity, which guarantees together with (58) that the spacetime is asymptotically flat.

In order to introduce the discontinuities in the metric tensor and in the electromagnetic four potential, we apply the transformation

$$z \to |z| + a\,, \tag{64}$$

with a a positive constant, to the function U_n.

Accordingly, we now have

$$U_n = -\sum_{l=0}^{n} \frac{C_l P_l\left((|z|+a)/R\right)}{R^{l+1}}\,, \tag{65}$$

with

$$R = \sqrt{r^2 + (|z|+a)^2}\,. \tag{66}$$

Let us consider the first term in equation (62),

$$U_0 = -\frac{C_0}{R}\,, \tag{67}$$

with R is given by (66). In consequence

$$\bar{\sigma} = \frac{\sigma}{\sigma_0} = \left(\frac{1+\bar{C}_0}{\bar{R}_0+\bar{C}_0}\right)^3\,, \tag{68}$$

$$\bar{p_r} = \frac{p_r}{p_{r_0}} = \frac{1}{\bar{R}_0}\left(\frac{1+\bar{C}_0}{\bar{R}_0+\bar{C}_0}\right)^3\,, \tag{69}$$

$$P_{(r)} = \frac{-2\sqrt{2}\bar{C}_0\bar{R}_0}{(\bar{R}_0+\bar{C}_0)^2\bar{r}}\,. \tag{70}$$

Here

$$\sigma_0 = \sigma\big|_{(r=0)}\,, \tag{71}$$

$$p_{r_0} = p_r\big|_{(r=0)}\,. \tag{72}$$

and we introduce dimensionless quantities through the relations

$$r = a\bar{r}\,, \tag{73}$$

$$C_0 = a\bar{C}_0\,, \tag{74}$$

$$R_0 = a\bar{R}_0\,. \tag{75}$$

We reduce the possible C_0 constants evaluating the agreement of the energy conditions,

$$\sigma \geq 0\,, \tag{76}$$

$$|\sigma| \geq |p_i|\,, \tag{77}$$

$$\sigma + p_i \geq 0\,, \tag{78}$$

$$\sigma + p_r + p_\phi + p_z \geq 0\,, \tag{79}$$

and found that they are satisfied only if $C_0 = a$.

We plot in figure 1 the dimensionless surface energy density $\bar{\sigma}$ and the dimensionless radial pressure $\bar{r}$ as functions of $\bar{r}$ for $\bar{C_0} = 1$.

From the plots, it can be seen that they are everywhere positive, decay when $\bar{r}$ increases and their maximum occurs at the center of the disk.

Also, in figure 1 we plot the dimensionless radial component of the electric polarization vector $\bar{P_r}$ as a function of $\bar{r}$.

We found it has a singularity at the center of the disk and it decays when the distance from the center increases.

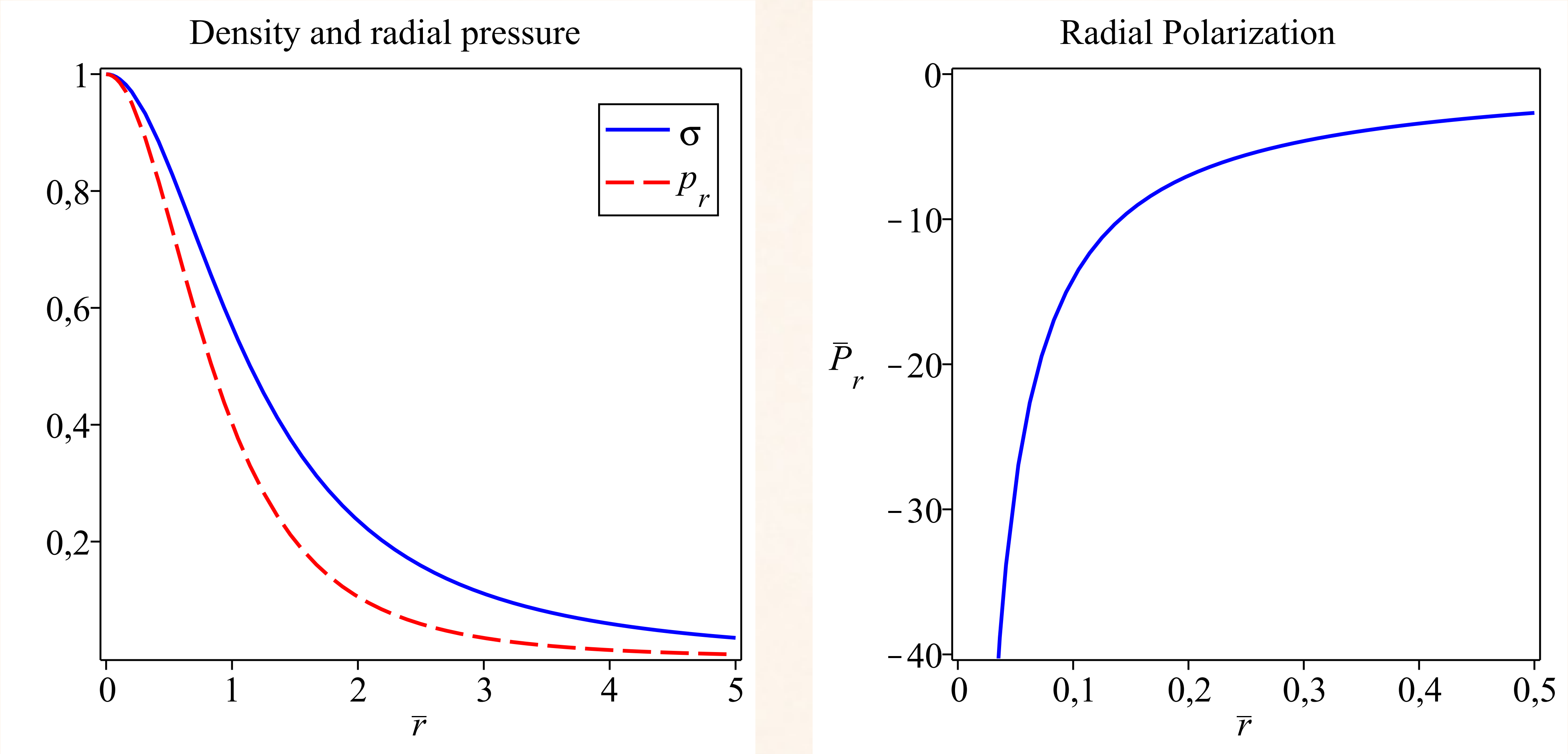

Figure 1: density $\bar{\sigma}$, radial pressure $\bar{p_r}$ and radial polarization $\bar{P}_r$ of a polarized disk in the model $n = 0$ with $\bar{C}_0 = 1$.