The propagation velocity of a monochromatic cylindrical gravito-electric wave

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1. Introduction

The Weyl conformal curvature tensor can be decomposed into two parts, given any reference frame, the gravito-electric and gravito-magnetic inhomogeneity fields. In the case of pure gravito-electric (magnetic) fields, it is possible to identify the frame for which we can completely eliminate the alternative gravito-magnetic (electric) field. In this reference frame the super-Poynting vector vanishes, this means that this frame is at rest with respect to the field and what is more, we can determine its velocity of propagation.

2. Invariants of the Weyl tensor.

A reference frame can be identified with the world line of an observer. The unit vector τ tangent to the world lines (monad vector) is the 4-velocity of the observer [3]. Given any reference frame, the electric and magnetic parts of the Weyl tensor

$$X_{\alpha\beta} = C_{\mu\alpha\nu\beta}\tau^{\mu}\tau^{\nu}, \quad Y_{\alpha\beta} = -C^{*}_{\mu\alpha\nu\beta}\tau^{\mu}\tau^{\nu} \tag{1}$$

are symmetric and traceless 3-tensors. The four independent invariants of the Weyl tensor

$$I_1 = C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}, \quad I_2 = C^{*}_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta},$$
$$I_3 = C_{\alpha\beta\gamma\delta}C^{\alpha\beta\epsilon\eta}C_{\epsilon\eta}{}^{\gamma\delta}, \quad I_4 = C^{*}_{\alpha\beta\gamma\delta}C^{\alpha\beta\epsilon\eta}C_{\epsilon\eta}{}^{\gamma\delta}.$$

can be rewritten in terms of X and Y

$$I_1 = 8(X_{\beta\delta}X^{\beta\delta} - Y_{\beta\delta}Y^{\beta\delta}), \quad I_3 = 16X^{\alpha}_{\beta}(X^{\beta}_{\gamma}X^{\gamma}_{\alpha} - 3Y^{\beta}_{\gamma}Y^{\gamma}_{\alpha}),$$
$$I_2 = -16X_{\beta\delta}Y^{\beta\delta}, \quad I_4 = 16Y^{\alpha}_{\beta}(Y^{\beta}_{\gamma}Y^{\gamma}_{\alpha} - 3X^{\beta}_{\gamma}X^{\gamma}_{\alpha}).$$

3. Bel-Robinson Tensor and Super-Poynting vector.

The superenergy tensor of Bel-Robinson is constructed with the Weyl tensor

$$T^{\alpha\beta\lambda\mu} \equiv C^{\alpha\rho\lambda\sigma}C^{\beta}{}_{\rho}{}^{\mu}{}_{\sigma} + {*C^{\alpha\rho\lambda\sigma}} * C^{\beta}{}_{\rho}{}^{\mu}{}_{\sigma}, \tag{2}$$

and the super-Poynting vector

$$\mathcal{P}_{\alpha} = b_{\alpha\gamma}T^{\gamma\beta\lambda\mu}\tau_{\beta}\tau_{\lambda}\tau_{\mu}, \tag{3}$$

where $b_{\alpha\gamma}$ projects onto the 3-subspace orthogonal to τ, can be rewritten after some calculations

$$\mathcal{P}_{\alpha} = 2b^{\varkappa}_{\alpha}X^{\lambda\nu}Y_{\gamma\nu}\tau_{\sigma}E^{\sigma\gamma}{}_{\varkappa\lambda}, \tag{4}$$

where $E^{\sigma\gamma}{}_{\varkappa\lambda}$ is the axial tensor of Levi-Civita.

4. Einstein-Rosen space-time.

Consider the line element in cylindrical coordinates

$$ds^2 = e^{2(K-U)}(dt^2 - d\rho^2) - e^{2U}dz^2 - e^{-2U}\rho^2 d\varphi^2. \tag{5}$$

Function $U(\rho, t)$ must satisfy the cylindrical wave equation

$$U'' + \rho^{-1}U' - \ddot{U} = 0. \tag{6}$$

A solution to this equation corresponding to a traveling monochromatic gravitational wave is given by

$$U = AJ_0(\omega\rho)\cos(\omega t) + AY_0(\omega\rho)\sin(\omega t).$$

where J_0 and Y_0 are Bessel functions of first and second kind respectively. The metric function $K(\rho, t)$ can be obtained from

$$K' = \rho(U'^2 + \dot{U}^2), \quad \dot{K} = 2\rho U'\dot{U} \quad \longrightarrow \quad K = \int \dot{K}dt + \int K'd\rho \tag{7}$$

The only non-vanishing invariants of the Weyl tensor are I_3 and

$$I_1 = \frac{16}{\rho^2}e^{4(U-K)}\Big(U'^2 - 6\rho^3U'U''\dot{U}^2 + 3\rho U'\dot{U}^2 + \rho U'U'' - 2\rho^3U''U'^3$$
$$+3\rho^2U''\dot{U}^2 + 3\rho^2U''U'^2 - 9\rho^2\dot{U}^2U'^2 + 3\rho^4U'^2\dot{U}^4 - 3\rho^4\dot{U}^2U'^4 - 3\rho^3U'\dot{U}^4$$
$$+6\rho^3\dot{U}^2U'^3 + 2\rho^3U'\dot{U}^3 + \rho^2U''^2 + 2\rho^2U'^4 + \rho^4U'^6 - 3\rho^3U'^5 + 3\rho^2\dot{U}^4 - \rho^2\dot{U}'^2$$
$$-\rho^4\dot{U}^6 - 6\rho^2\dot{U}U'\dot{U}' + 6\rho^3\dot{U}\dot{U}'U'^2\Big).$$

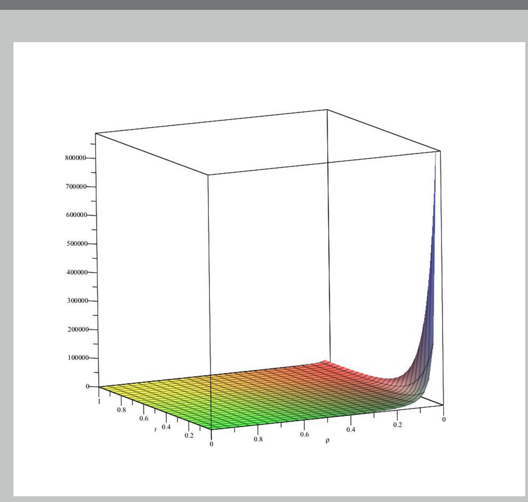

Figure 1: Invariant $I_1(\rho, t)$ for the monochromatic wave.

Propagation velocity of the gravito-electric wave.

Using the frame

$$\tau = \tau^{(0)}X_{(0)} + \tau^{(1)}X_{(1)}, \tag{8}$$

it can be found the only non-trivial component of the gravito-magnetic field

$$Y_{(2)(3)} = M(\rho, t)[(\tau^{(0)})^2 + (\tau^{(1)})^2] + N(\rho, t)\tau^{(0)}\tau^{(1)}, \tag{9}$$

where

$$M(\rho, t) := \left(3\rho\dot{U}U'^2 + \rho\dot{U}^3 - \dot{U}' - 3\dot{U}U'\right)e^{2(U-K)}$$
$$N(\rho, t) := \left(6\rho U'\dot{U}^2 + 2\rho U'^3 - 3\dot{U}^2 - 3U'^2 - 2U'' - \rho^{-1}U'\right)e^{2(U-K)}.$$

Since $\tau \cdot \tau = \left[\tau^{(0)}\right]^2 - \left[\tau^{(1)}\right]^2 = 1$, then we can choose

$$\tau^{(0)} = \cosh(\psi), \qquad \tau^{(1)} = \mathrm{senh}(\psi).$$

In the comoving frame with the gravito-electric wave, we can eliminate the component $Y_{(2)(3)}$ if I_1 is positive definite. This yields

$$\frac{\tanh\psi}{1 + \tanh^2\psi} = -\frac{M(\rho, t)}{N(\rho, t)}.$$

Clearly in this frame, the super-Poynting vector (4) vanishes. Now we indroduce another frame $\tilde{\tau} = X_{(0)}$, and its 3-velocity relative to the comoving frame τ

$$v_{(\mu)} = \frac{1}{\tau \cdot \tilde{\tau}}\tilde{b}_{(\mu)(\nu)}\tau^{(\nu)} = \frac{1}{\tau \cdot \tilde{\tau}}\left[g_{(\mu)(\nu)} - \tilde{\tau}_{(\mu)}\tilde{\tau}_{(\nu)}\right]\tau^{(\nu)}$$
$$= \frac{1}{\tau \cdot \tilde{\tau}}\left[\tau_{(\mu)} - (\tau \cdot \tilde{\tau})\tilde{\tau}_{(\mu)}\right] = -\tanh(\psi)\delta^{1}_{\mu},$$

Finally, we can conclude

$$0 \leq \frac{v}{1 + v^2} = \left|-\frac{M(t, \rho)}{N(t, \rho)}\right| \leq \frac{1}{2}. \tag{10}$$

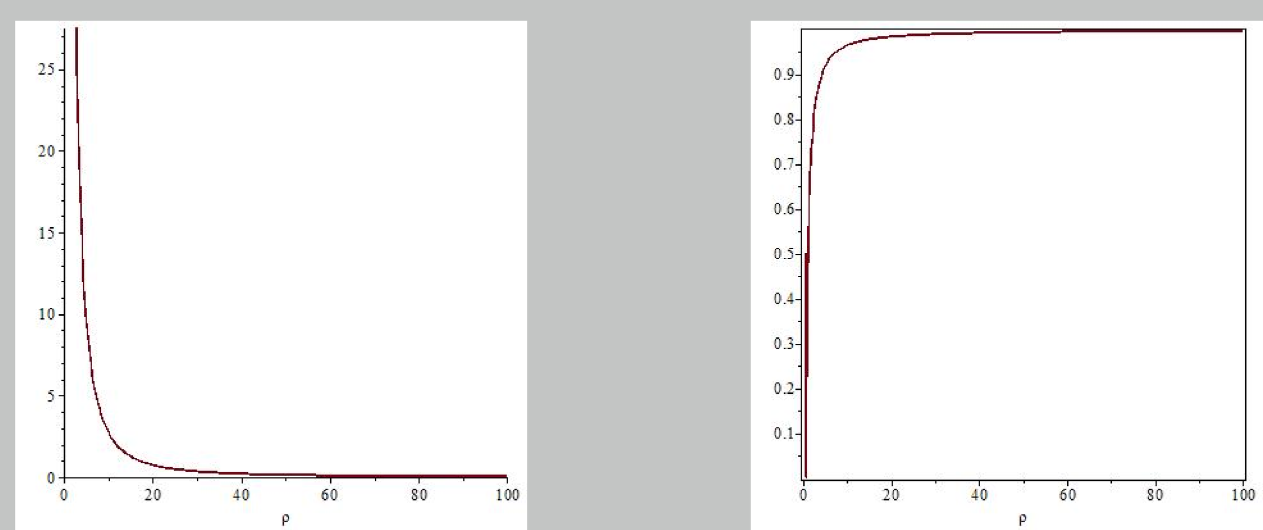

Figure 2: Invariant $I_1(\rho + 1, \rho)$ *(left)*. Propagation velocity magnitude $v(\rho + 1, \rho)$ *(right)*.

Concluding remarks

We have found the frame for which we can completely eliminate the gravito-magnetic part of the Einstein-Rosen monochromatic wave, therefore the Super-Poynting vector vanishes as well in this frame. This fact shows the existence of a comoving frame with the gravito-electric part that propagates at sub-luminal speed.

References

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