The relativistic kinetic theory of gases with applications to black hole accretion

70 & 70 Fiesta de Gravitación Clásica y Cuántica:

Encuentro con dos Maestros de la Física Teórica de América Latina

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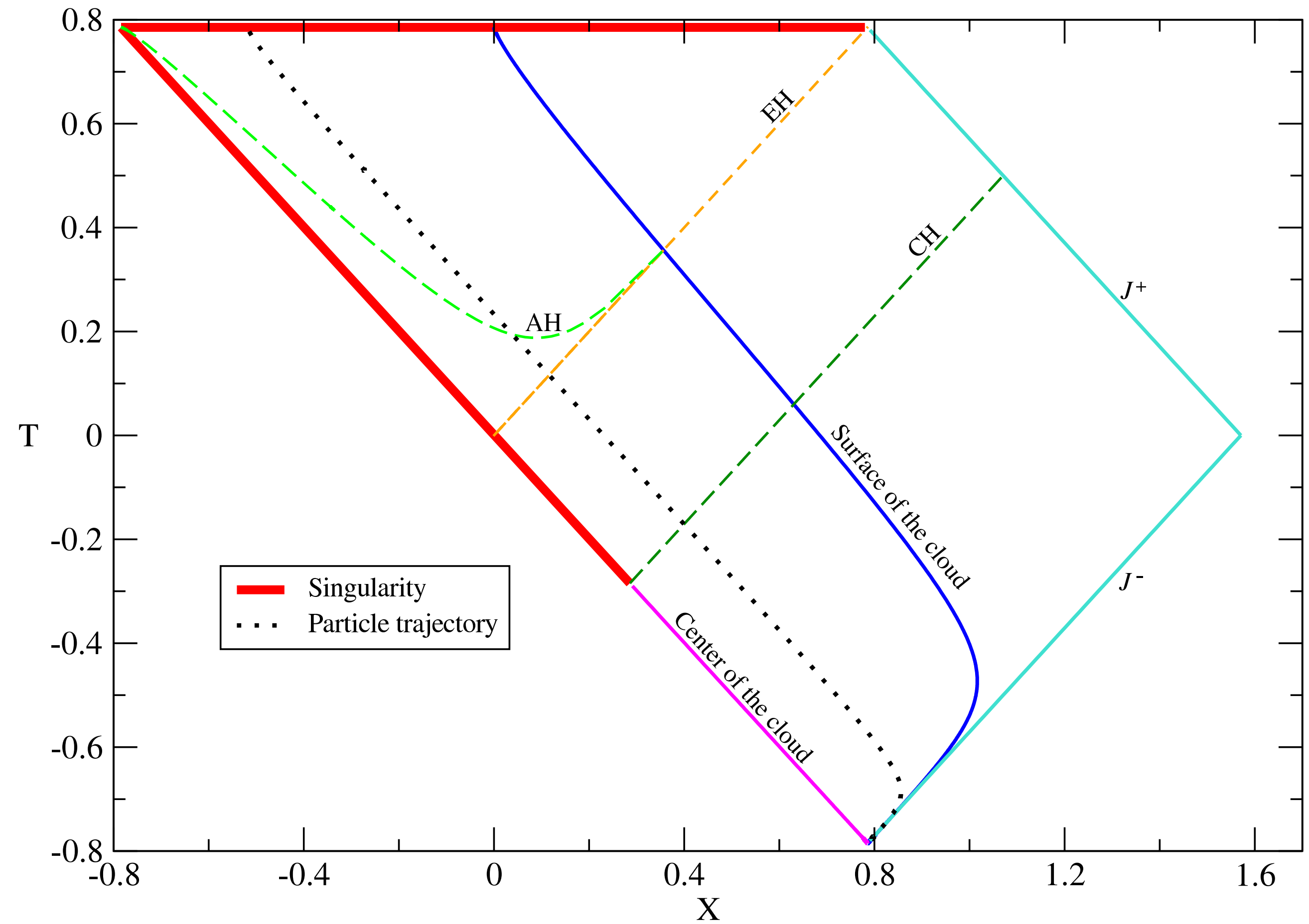
Outline

- Motivation
- Cotangent bundle and Hamiltonian structure
- Most general collisionless distribution function on a Schwarzschild background
- Spherical steady-state solutions
- Stability
- Conclusions and outlook

Relativistic kinetic theory: Motivation

- Description of dilute, relativistic gas in the presence of strong gravitational field.
- Accretion into a black hole
- Distribution of stars around supermassive black holes (in this case collisionless).
- Cosmic Censorship Hypothesis: Provide a description of the matter that goes beyond the scalar field model or the traditional perfect fluids and magneto-fluids.



Dust collapse (Eardley & Smarr, Christodoulou, Newman…)
Shown here a conformal diagram from N. Ortiz & OS 2011

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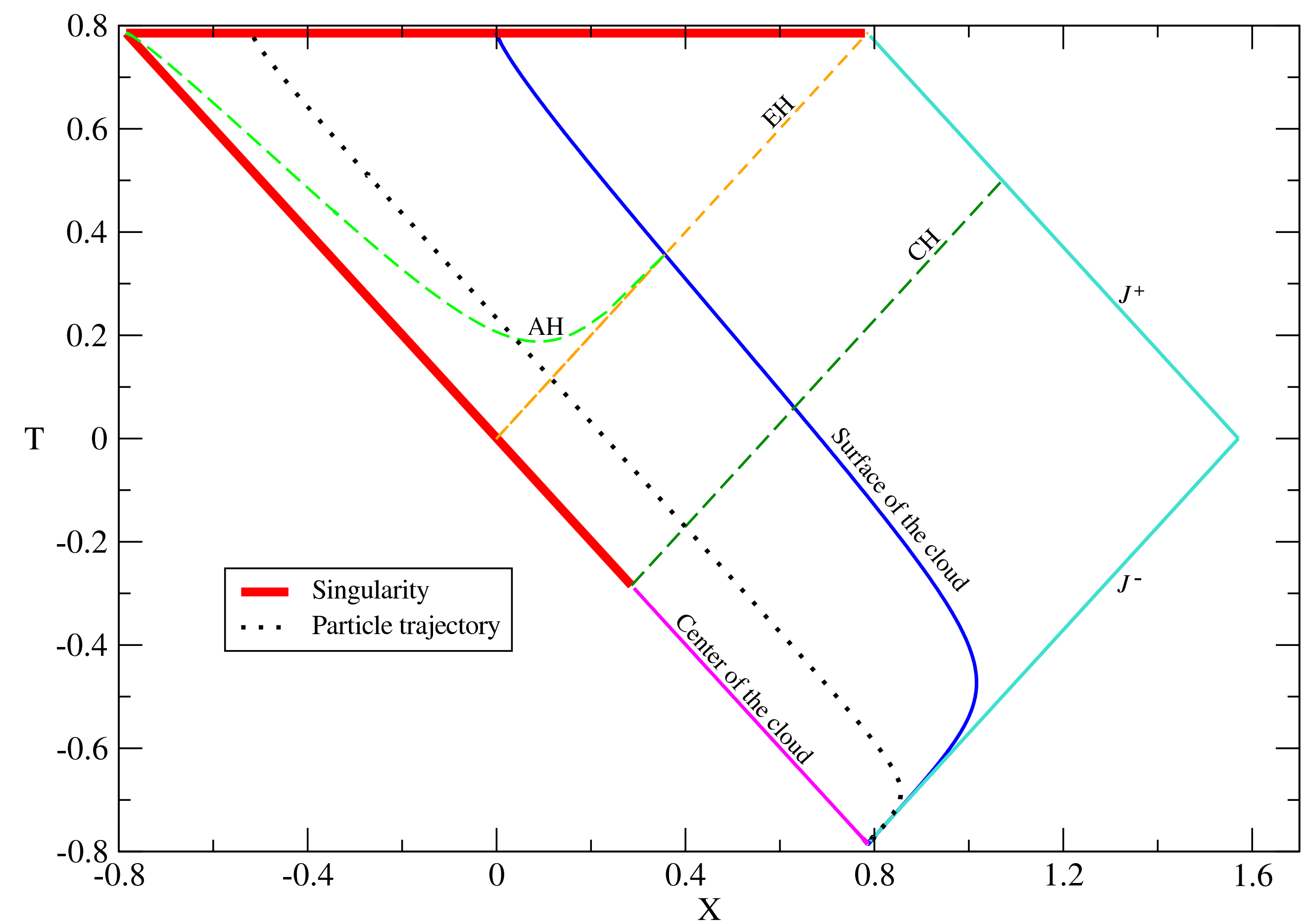

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General covariance requires *geometric formulation* of the theory!

Relativistic kinetic theory: Motivation

- Previous work regarding the formulation of relativistic kinetic theory: Jüttner, Synge, Taubner & Weinberg, Israel, Lindquist, Ehlers, Cercignani & Kremer, …
- Mathematical questions (well-posedness, global existence, static and stationary solutions): Rein & Rendall, Andréasson, Kunze, Dafermos, Ringström, Fajman, Joudioux, Smulevici, …

In this work we start with a simple case: A collisionless, relativistic gas propagating on a Schwarzschild black hole background.

Collaborators: Paola Rioseco, Thomas Zannias (IFM)

Geometry of the cotangent bundle

- Spacetime manifold: (M, g)
- Relativistic phase space (cotangent bundle):

  $T^*M = \{(x,p) : x \in M, p \in T^*_x M\}$

- Natural projection map:

  $\pi : T^*M \to M, (x,p) \mapsto x$

- Poincaré (or canonical) one-form:

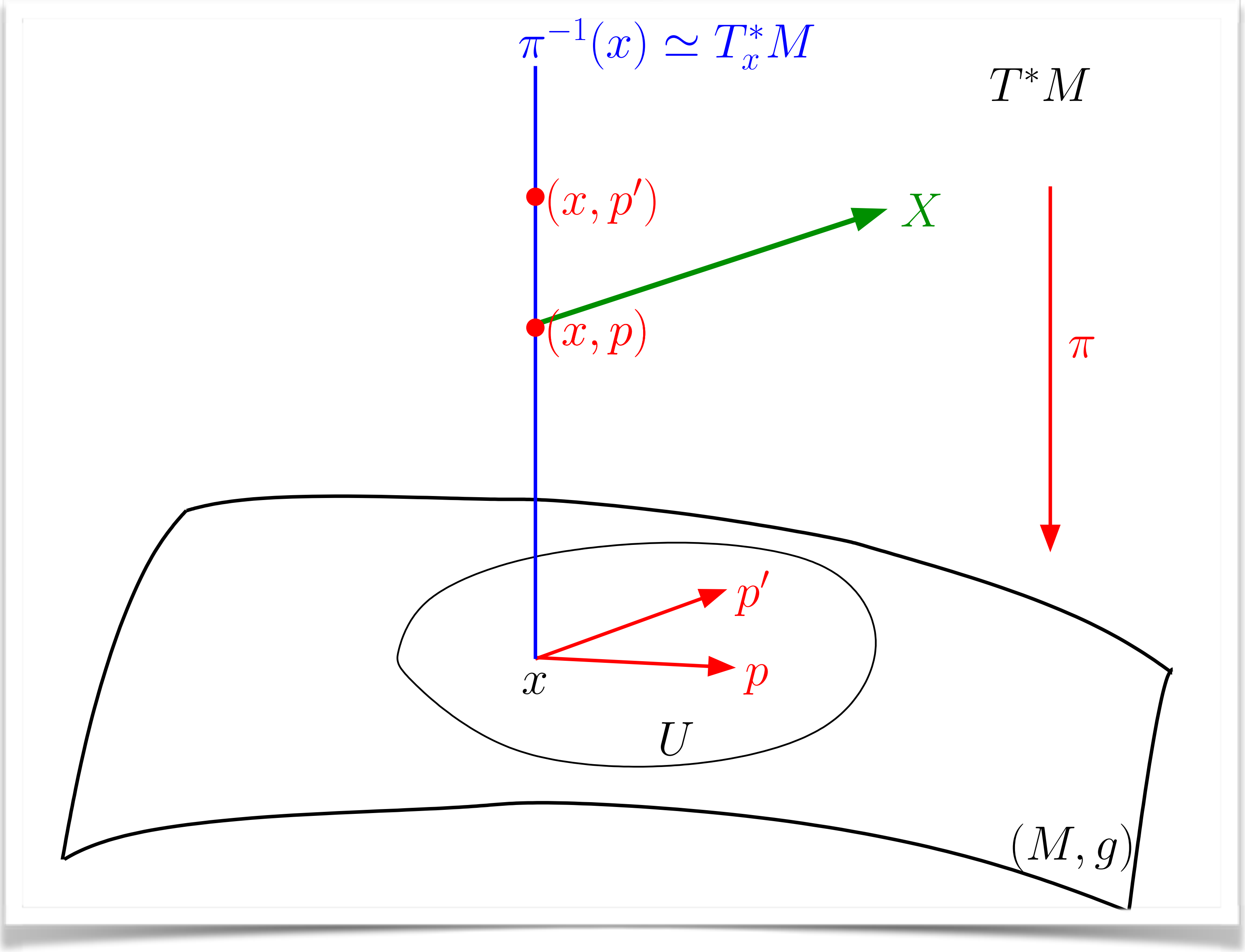
  $\Theta_{(x,p)}(X) = p(\pi_{*x}(X))$

- Symplectic form:

  $\Omega = d\Theta = dp_\mu \wedge dx^\mu$

- Free particle Hamiltonian:

  $H(x,p) := \frac{1}{2} g_x^{-1}(p,p) = \frac{1}{2} g^{\mu\nu}(x) p_\mu p_\nu$



Geometry of the cotangent bundle

- Liouville vector field is defined as corresponding Hamiltonian vector field:

$$dH = -i_L \Omega = \Omega(\cdot, L)$$

- Explicitly:

$$L = g^{\mu\nu}(x) p_\nu \frac{\partial}{\partial x^\mu} - \frac{1}{2} p_\alpha p_\beta \frac{\partial g^{\alpha\beta}(x)}{\partial x^\mu} \frac{\partial}{\partial p_\mu}$$

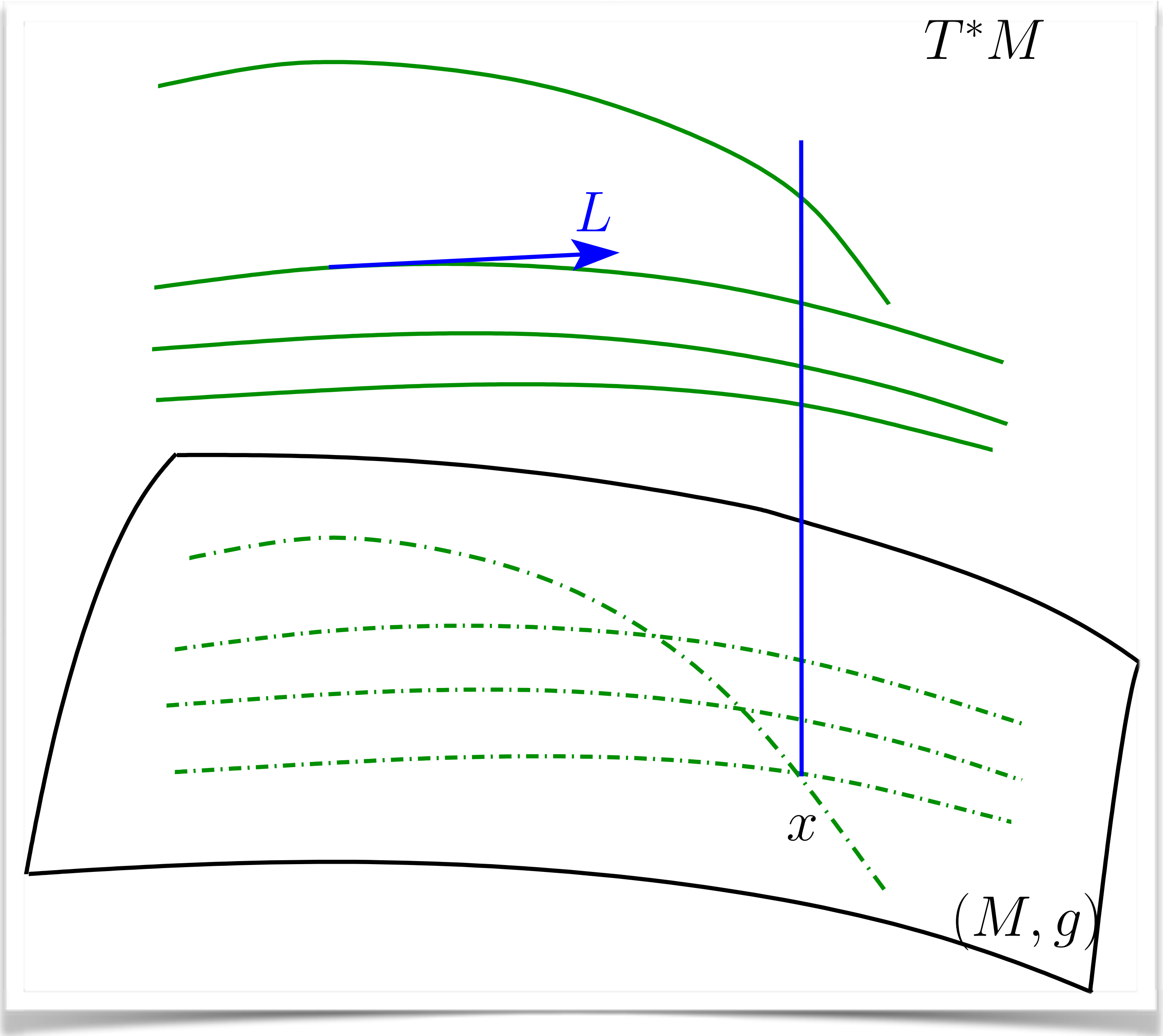
- Relativistic Boltzmann equation:

$$L[f] = Q(f, f)$$

collision term

one-particle distribution function

- In the collisionless case this reduces to the Liouville equation.



Solving the Liouville equation

- Assume a Schwarzschild background (mass = 1) in ingoing Eddington-Finkelstein coordinates (also works for Kerr!). The free-particle Hamiltonian is

$$H(x,p) = \frac{1}{2}\left[-\left(1+\frac{2}{r}\right)p_t^2 + \frac{4}{r}p_t p_r + \left(1-\frac{2}{r}\right)p_r^2 + \frac{1}{r^2}\left(p_\vartheta^2 + \frac{p_\varphi^2}{\sin^2\vartheta}\right)\right]$$

- Conserved quantities: rest mass (m), energy (E), angular momentum

- Integrable Hamiltonian system, so can apply standard tools from dynamical systems. Invariant subsets:

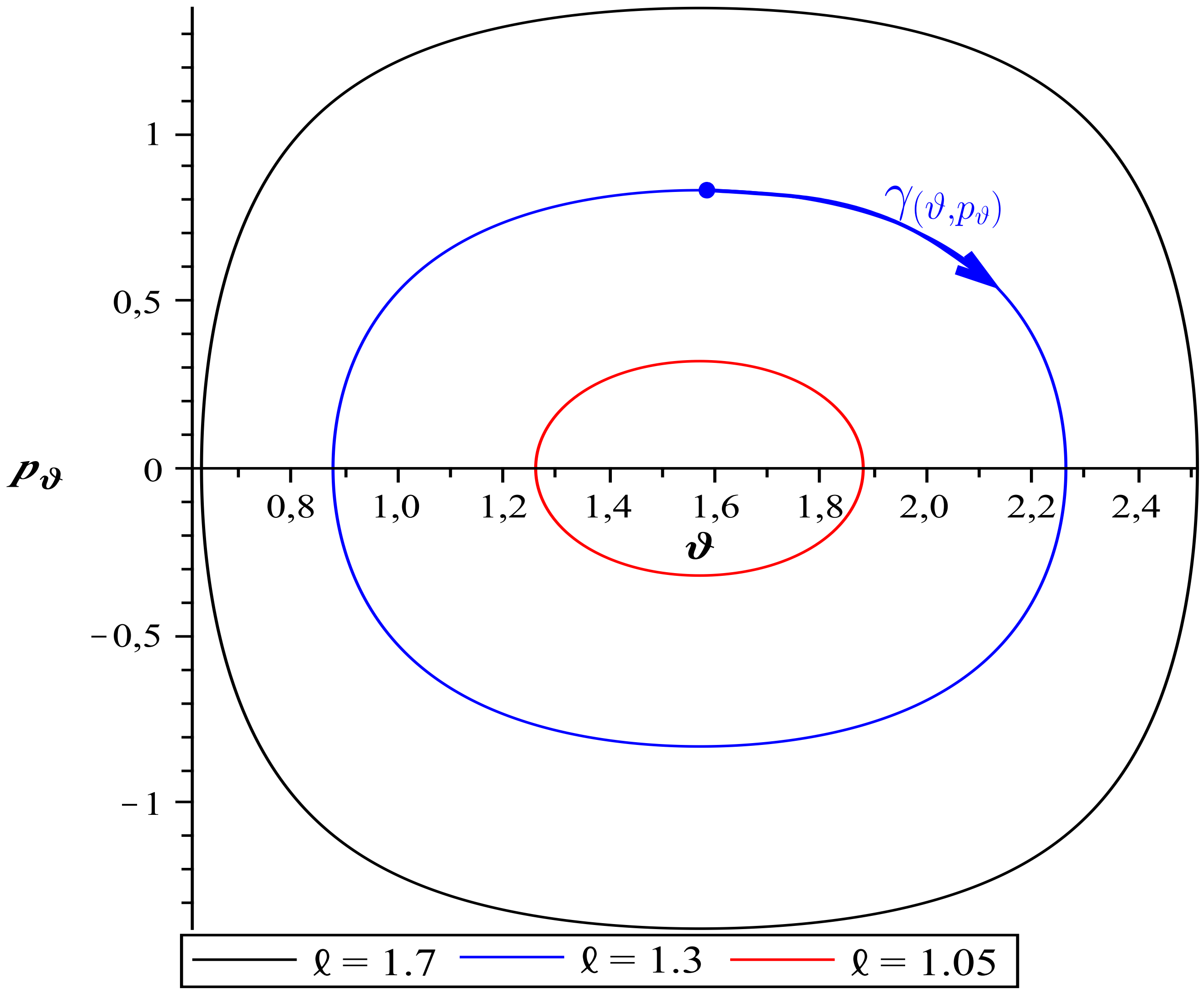
$$\Gamma_{m,E,\ell_z,\ell} := \left\{(x,p) \in T^*M : H(x,p) = -\frac{1}{2}m^2, p_t = -E, p_\varphi = \ell_z, p_\vartheta^2 + \frac{p_\varphi^2}{\sin^2\vartheta} = \ell^2\right\}$$

- By definition, these sets are invariant with respect to the Hamiltonian flow, and restriction of Poincaré one-form to these sets is closed. If not empty, these are 4d submanifolds of phase space which are topologically equal to

$\mathbb{R} \times S^1 \times S^1 \times S^1$ or $\mathbb{R}^2 \times S^1 \times S^1$

Solving the Liouville equation

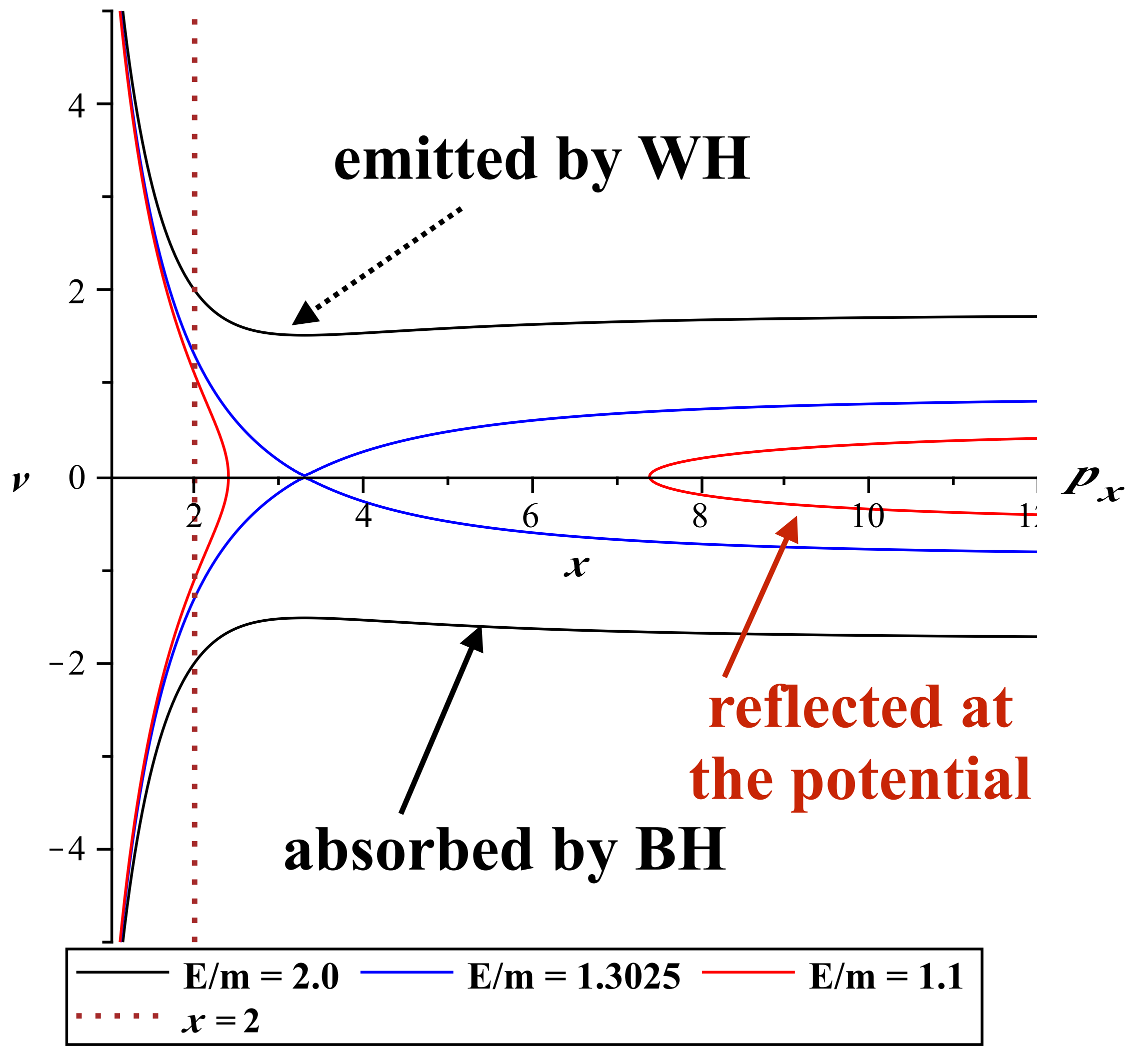
- (t, φ) are free, and give a $\mathbb{R} \times S^1$ factor.
- $p_\vartheta^2 + \dfrac{\ell_z^2}{\sin^2 \vartheta} = \ell^2$

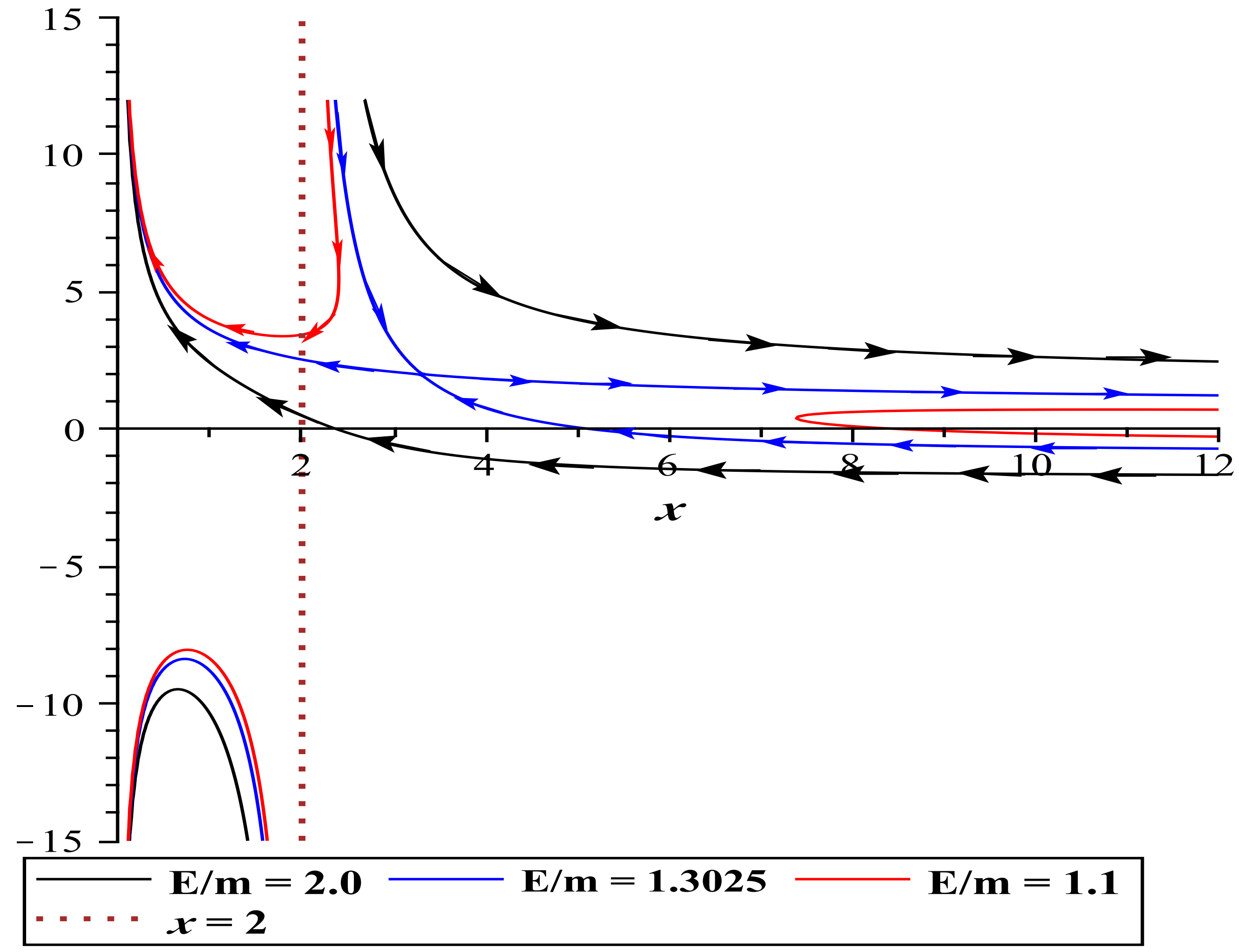


Solving the Liouville equation

- Motion in the radial direction:

$$\left[\left(1-\frac{2}{r}\right)p_r-\frac{2}{r}E\right]^2+V_{m,\ell}(r)=E^2 \text{ with } V_{m,\ell}(r)=\left(1-\frac{2}{r}\right)\left(m^2+\frac{\ell^2}{r^2}\right)$$





$x=r, \quad v=\dot{r}$

Solving the Liouville equation

- New symplectic coordinates defined by generating function

$$S(x; m, E, \ell_z, \ell) = \int_{\gamma_x} \Theta = \int_{\gamma_x} p_\mu dx^\mu$$

- $P_0 = m^2/2,\ P_1 = E,\ P_2 = \ell_z,\ P_3 = \ell^2$
  $Q^\mu := \frac{\partial S}{\partial P_\mu}$ (Q^2 and Q^3 are angles)
- In terms of the new coordinates the Hamiltonian is simply $H = -P_0$ and hence the Liouville equation becomes trivial:

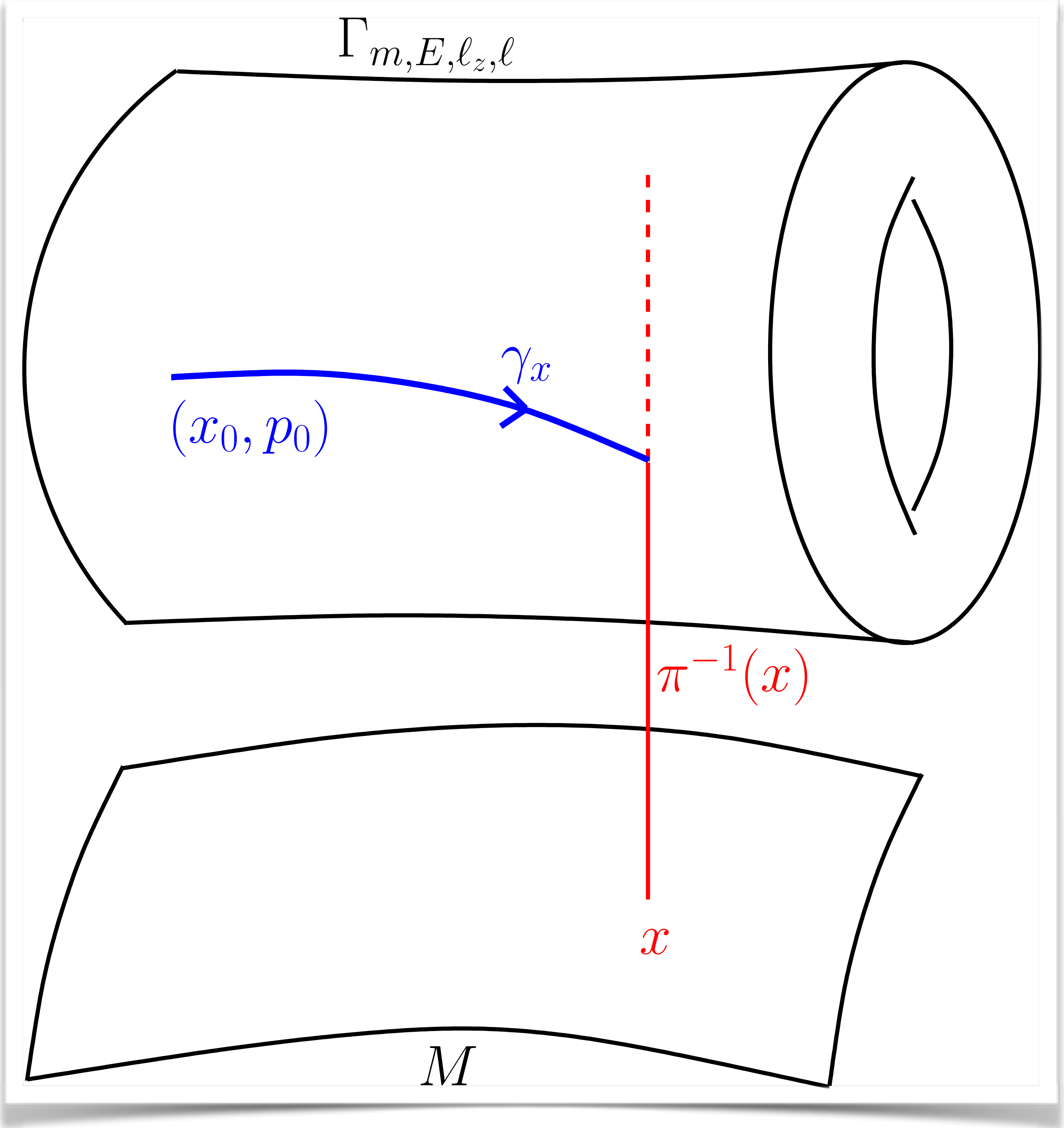
$$L[f] = -\frac{\partial f}{\partial Q^0} = 0$$

- Most general solution:

$$f(x,p) = F(Q^1, Q^2, Q^3, P_0, P_1, P_2, P_3)$$

t + ... $\quad \varphi + \dots$

- Stationary and axisymmetric if independent of the first two variables.



Spherical steady-state solutions

- For a spherical, steady-state solution:

$$f(x,p) = F(m, E, \ell)$$

- Assume, further, a simple gas which is in equilibrium at infinity:

$$f(x,p) = \alpha\delta\left(\sqrt{-2H(x,p)} - m\right) e^{-\beta E}, \quad \beta\text{: inverse temperature}$$

- Compute the observables: current density and stress energy-momentum tensor:

$$J_\mu(x) = \int\limits_{\pi^{-1}(x)} p_\mu f(x,p) \frac{d^4p}{\sqrt{-g}}, \qquad T_{\mu\nu}(x) = \int\limits_{\pi^{-1}(x)} p_\mu p_\nu f(x,p) \frac{d^4p}{\sqrt{-g}}$$

- Decomposition

$$J^\mu = n u^\mu, \qquad T^{\mu\nu} = \varepsilon e_0^\mu e_0^\nu + \sum_{j=1,2,3} p_j e_j^\mu e_j^\nu$$

particle density **energy density** **principle pressures**

In general, e_0^μ is not proportional to u^μ, and $p_1 \neq p_2 \neq p_3$.

Spherical steady-state solutions

- Observables have three contributions:

$$T_{\mu\nu} = T_{\mu\nu}^{(in)} + T_{\mu\nu}^{(scat)} + T_{\mu\nu}^{(bounded)}$$

absorbed by BH
no contribution at infinity

scattered particles
no contribution at horizon

bounded trajectories
no contribution at horizon nor infinity

- At infinity, gas behaves like isotropic fluid with ideal gas equation of state $(z = m\beta)$

$$n_\infty(z) = 4\pi\alpha m^4 \frac{K_2(z)}{z}, \qquad \varepsilon_\infty(z) = 4\pi\alpha m^5 \left[\frac{K_1(z)}{z} + \frac{3K_2(z)}{z^2}\right], \qquad p_\infty(z) = \beta^{-1} n_\infty(z)$$

- At the horizon, the gas ceases to be isotropic. For high z:

$$\frac{\varepsilon_H}{n_H} = \frac{mc^2}{2\sqrt{3}}\left(3 + \sqrt{\frac{31}{3}}\right) \simeq 1.79mc^2$$

$$\frac{p_{rad}}{n_H} = \frac{mc^2}{2\sqrt{3}}\left(-3 + \sqrt{\frac{31}{3}}\right) \simeq 0.062mc^2, \qquad \frac{p_{tan}}{n_H} = \frac{mc^2}{4\sqrt{3}} \simeq 0.144mc^2$$

The tangential pressure is about twice as large as the radial one!

Spherical steady-state solutions

- Accretion rate for large z:

$$\mu = \frac{4\pi J^r}{n_\infty} \simeq -16M^2\sqrt{2\pi z}$$

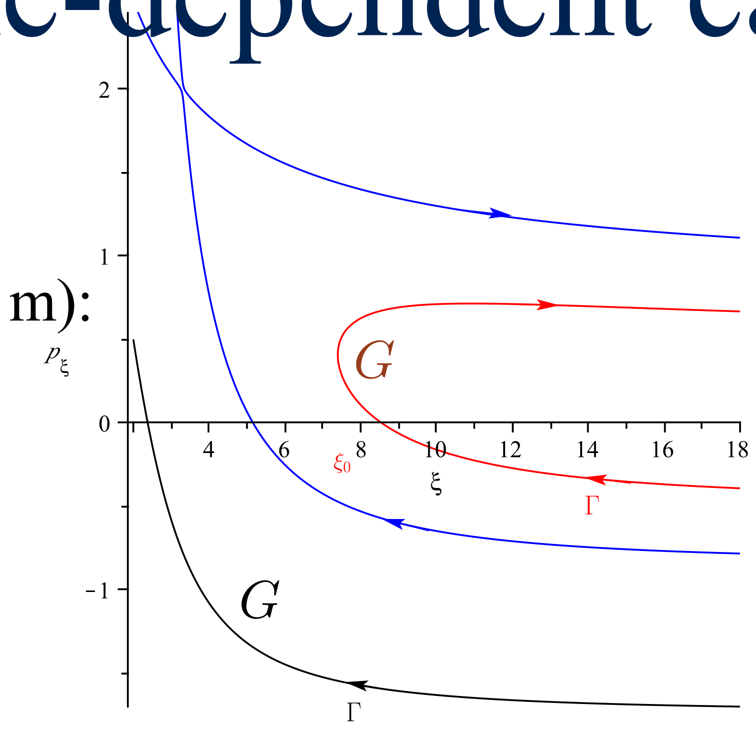
- Compression rate for large z:

$$\frac{n_H}{n_\infty} \simeq \sqrt{\frac{6z}{\pi}}$$

- Both are smaller by a factor of $z = m\beta$ compared to the hydrodynamic case (Bondi accretion). For accretion from the interstellar medium, $z \simeq 10^9$
(see book by Shapiro & Teukolsky and references therein for corresponding Newtonian-based calculation)

- Interpretation (?): When collisions are taken into account, particles with large angular momentum will collide and augment radial pressure.

Stability result for time-dependent case



- The general solution takes the form (for fixed mass m):

$$f(x,p) = F(G-t, Q^2, Q^3, E, \ell_z, \ell^2)$$

with G independent of t.

- Suppose the initial distribution function satisfies the following conditions:
(1) $F = 0$ for bounded orbits
(2) $0 \le F \le \alpha e^{-\beta E}$ (i.e. bounded by an equilibrium distribution function)
(3) $\lim_{G\to\pm\infty} F(G, Q^2, Q^3, E, \ell_z, \ell^2) = f_\infty(E)$ uniformly in $(Q^2, Q^3, \ell_z, \ell^2)$

Then, along the world line of static observers, the current density and stress energy-momentum converge pointwise to the corresponding observables of the steady-state solution with distribution function $f_\infty(E)$

- In particular, if $f_\infty(E) = 0$ the gas disperses completely.

Conclusions and Outlook

- Rich structure of the cotangent bundle leads naturally to symplectic structure and many other nice properties no mentioned in this talk (bundle metric, volume form, …)
- When geodesic motion on spacetime manifold is integrable, one can introduce "good" symplectic coordinates on relativistic phase space, which trivialize the Liouville vector field (action-angle-like variables).
- Using these coordinates, one can study the behavior of observables "by inspection", and prove stability results, for instance.
- In future work, we want to study disk solutions around Kerr black holes (in this case Carter constant replaces the total angular momentum)
- The use of "good" symplectic coordinates might be useful for the study of perturbed systems, for example when taking into account collisions or the self-gravity of the gas at the perturbative level.